Math 324 Quiz 4 Practice

February 23, 2017

Problem 1. Write the definition of a vector field $\vec{F}(x, y)$ being conservative.

Solution 1. A vector field $\vec{F}(x, y)$ is conservative if there exists a function $f(x, y)$ satisfying $\vec{F}(x, y) = \nabla f$.

Problem 2. Give an example of a conservative vector field.

Solution 2. An example is $\vec{F}(x, y) = \langle 2x, 2y\rangle$ because $\vec{F} = \nabla(x^2 + y^2)$.

Problem 3. Give an example of a non-conservative vector field.

Solution 3. An example is $\vec{F}(x, y) = \langle y, 0\rangle$ because

$$\frac{\partial}{\partial x}0 \neq \frac{\partial}{\partial y}y.$$

Problem 4. Consider the curve C parametrized by

$$\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle$$

as $0 \leq t \leq 10\pi$. Calculate the integral $\int_C \vec{F} \cdot d\vec{s}$ for the vector field

$$\vec{F}(x, y) = \langle x + 2xy, y + x^2\rangle.$$

Solution 4. We will present two different solutions to this problem, one using the Fundamental Theorem of Path Integrals, and one using independence of path for line integrals of conservative vector fields.
Solution using Fundamental Theorem: The vector field $\vec{F}(x, y)$ is conservative, since

$$\frac{\partial}{\partial x}(y + x^2) = 2x = \frac{\partial}{\partial y}(x + 2xy).$$

Therefore we can find $f(x, y)$ such that $\vec{F} = \nabla f$. This says that

$$f = \int f_x \partial x = \int (x + 2xy)\partial x = \frac{1}{2}x^2 + x^2y + h(y).$$

From this we calculate $f_y = x^2 + h'(y)$, and since $f_y = y + x^2$ this says that $h'(y) = y$. Hence $h(y) = \frac{1}{2}y^2 + B$ for some constant B Thus

$$f(x, y) = \frac{1}{2}x^2 + x^2y + \frac{1}{2}y^2 + B.$$

The starting point of C is $\vec{r}(0) = (0,0)$, and the ending point is $\vec{r}(0) = (10\pi, 0)$. By the Fundamental Theorem of Line Integrals we get

$$\int_C \vec{F} \cdot d\vec{s} = f(10\pi, 0) - f(0,0) = \frac{1}{2}(10\pi)^2 + B - B = 50\pi^2.$$

Solution using Path Independence: The vector field $\vec{F}(x,y)$ is conservative, since

$$\frac{\partial}{\partial x}(y + x^2) = 2x = \frac{\partial}{\partial y}(x + 2xy).$$

Therefore the value of $\int_C \vec{F} \cdot d\vec{s}$ depends only on $\vec{F}$ and the starting and ending points of the path C. The starting point of C is $\vec{r}(0) = (0,0)$, and the ending point is $\vec{r}(0) = (10\pi, 0)$. Therefore instead of integrating along C, we can integrate along the curve $\widetilde{C}$ parametrized by

$$\vec{\tilde{r}}(t) = \langle t, 0 \rangle, \ 0 \leq t \leq 10\pi,$$

since it has the same starting and ending points as the curve C. We calculate

$$\int_{\widetilde{C}} \vec{F} \cdot d\vec{s} = \int_0^{10\pi} \vec{F}(\vec{\tilde{r}}(t))\vec{\tilde{r}}'(t)dt = \int_0^{10\pi} \langle t, t^2 \rangle \cdot \langle 1, 0 \rangle dt = \int_0^{10\pi} t dt = 50\pi^2.$$

Problem 5. Let C be the closed curve formed by a circle of radius 1 centered at the origin. Calculate the integral

$$\oint_C \vec{F}(x,y) \cdot d\vec{s}$$

where here $\vec{F}(x,y)$ is the vector field

$$\vec{F}(x,y) = \langle y^3, -x^3 \rangle.$$

[Hint: Green's theorem makes the work easier]

Solution 5. We calculate

$$\omega = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3x^2 - 3y^2$$

and therefore by Green's theorem

$$\oint_C \vec{F} \cdot d\vec{s} = \int \int_R \omega dA$$

where R is the region inside C, ie. the disk of radius 1 centered at the origin. Using polar, we see $\omega = -3r^2$ and therefore

$$\int\int_R \omega dA = \int_0^{2\pi}\int_0^1 -3r^2 r dr d\theta = -2\pi.$$