Math 324 Quiz 4

March 7, 2017

Problem 1.

1. Write the definition of a vector field being conservative.
2. Write the definition of a vector field being non-rotational.
3. What is the relationship between torsion-free vector fields and non-rotational vector fields?

Solution 1.

1. A vector field $\vec{F}$ is conservative if there exists a scalar function f satisfying $\vec{F} = \nabla f$.
2. A vector field is non-rotational if its curl is zero. In two dimensions, by this we mean
$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$
3. For differentiable vector fields on simply connected domains, being conservative and being non-rotational are equivalent conditions.

Problem 2. Consider the vector field

$$\vec{F}(x, y) = \langle 1 + \cos(x + y), \cos(x + y) - 2y \rangle.$$

Show that $\vec{F}(x, y)$ is conservative.

Solution 2. We calculate

$$\frac{\partial F_2}{\partial x} = -\sin(x + y)$$
$$\frac{\partial F_1}{\partial y} = -\sin(x + y)$$

so the field is conservative.

Problem 3. Consider the vector field $\vec{F}(x, y)$ of the previous problem. Evaluate the integral $\int_C \vec{F} \cdot d\vec{s}$ where C is the curve defined by the parametric equation

$$\vec{r}(t) = \langle t^2, t\sin(t) \rangle, \quad 0 \le t \le 10\pi.$$

Solution 3. The starting point of the curve is $(0,0)$, while the ending point is $(100\pi^2, 0)$. Furthermore, we have $\vec{F}(x,y) = \nabla f(x,y)$ for $f(x,y) = x + \sin(x+y) + y^2$. Therefore by the fundamental theorem of line integrals

$$\int_C \vec{F} \cdot d\vec{s} = f(100\pi^2, 0) - f(0,0) = 100\pi^2 + \sin(100\pi^2).$$