Math 324 Quiz 5 Practice Solutions

February 28, 2017

Problem 1. Suppose that $f(x, y, z)$ is a scalar function and that $\vec{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$ is a three-dimensional vector field. Show that

$$\nabla \cdot (f\vec{F}) = f\nabla \cdot \vec{F} + \vec{F} \cdot \nabla f.$$

Solution 1. We calculate

$$\begin{aligned}
\nabla \cdot (f\vec{F}) &= \nabla \cdot \langle fF_1, fF_2, fF_3\rangle \\
&= (fF_1)_x + (fF_2)_y + (fF_3)_z \\
&= f_xF_1 + fF_{1x} + f_yF_2 + fF_{2y} + f_zF_3 + fF_{3z} \\
&= f_xF_1 + f_yF_2 + f_zF_3 + fF_{1x} + fF_{2y} + fF_{3z} \\
&= \vec{F} \cdot \nabla f + f\nabla \cdot \vec{F}.
\end{aligned}$$

Problem 2. Suppose that $\vec{F}(x, y, z)$ is a three-dimensional vector field. Show that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \Delta\vec{F},$$

where here

$$\Delta\vec{F} = \langle \Delta F_1, \Delta F_2, \Delta F_3\rangle ,$$

for $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ the Laplacian operator.

Solution 2. We calculate

$$\begin{aligned}
\nabla \times (\nabla \times \vec{F}) &= \nabla \times \langle F_{3y} - F_{2z}, F_{1z} - F_{3x}, F_{2x} - F_{1y}\rangle \\
&= (F_{2xy} - F_{1yy} - F_{1zz} + F_{3xz})\widehat{i} \\
&+ (F_{3yz} - F_{2zz} - F_{2xx} + F_{1yx})\widehat{j} \\
&+ (F_{1zx} - F_{3xx} - F_{3yy} + F_{2zy})\widehat{k}
\end{aligned}$$

Furthermore

$$\begin{aligned}
\Delta\vec{F} &= (F_{1xx} + F_{1yy} + F_{1zz})\widehat{i} \\
&+ (F_{2xx} + F_{2yy} + F_{2zz})\widehat{j} \\
&+ (F_{3xx} + F_{3yy} + F_{3zz})\widehat{k}
\end{aligned}$$

and

$$
\begin{aligned}
\nabla(\nabla \cdot \vec{F}) & \\
&= \nabla(F_{1x} + F_{2y} + F_{3z}) \\
&= (F_{1xx} + F_{2yx} + F_{3zx})\widehat{i} \\
&+ (F_{1xy} + F_{2yy} + F_{3zy})\widehat{j} \\
&+ (F_{1xz} + F_{2yz} + F_{3zz})\widehat{k}
\end{aligned}
$$

so that

$$
\begin{aligned}
\nabla(\nabla \cdot \vec{F}) - \Delta\vec{F} &= (F_{2yx} - F_{1yy} + F_{3zx} - F_{1zz})\widehat{i} \\
&+ (F_{1xy} - F_{2xx} + F_{3zy} - F_{2zz})\widehat{j} \\
&+ (F_{1xz} - F_{3xx} + F_{2yz} - F_{3yy})\widehat{k}
\end{aligned}
$$

Comparing $\nabla \times (\nabla \times \vec{F})$ and $\nabla(\nabla \cdot \vec{F}) - \Delta\vec{F}$, we see that they are the same.

Problem 3. For each value of $\vec{F}$, determine whether or not $\vec{F}$ is conservative. If it is, find a function f such that $\vec{F} = \nabla f$.

(a) $\vec{F}(x, y, z) = \langle 3xy^2z^2, 2x^2yz^3, 3x^2y^2z^2 \rangle$

(b) $\vec{F}(x, y, z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$

Solution 3.

(a) We calculate the curl of $\vec{F}$ to be

$$\nabla \times \vec{F} = \langle 0, 6xy^2z - 6xy^2z^2, 4xyz^3 - 6xyz^2 \rangle.$$

Since the curl of $\vec{F}$ is not zero, $\vec{F}$ is not conservative.

(b) We calculate the curl of $\vec{F}$ to be

$$\nabla \times \vec{F} = \langle 0, 0, 0 \rangle.$$

Therefore there exists f such that $\vec{F} = \nabla f$. This implies that

$$f = \int f_x \partial x = \int F_1 \partial x = \int e^{yz} \partial x = xe^{yz} + h(y, z).$$

From this we see

$$f_y = xze^{yz} + h_y(y, z), \quad f_z = xye^{yz} + h_z(y, z).$$

Since $f_y = F_2$ and $f_z = F_3$, this tells us

$$xze^{yz} + h_y(y, z) = xze^{yz}, \quad xye^{yz} + h_z(y, z) = xye^{yz}.$$

Therefore $h_y(y, z) = 0$ and $h_z(y, z) = 0$, so that $h(y, z) = C$ for some constant C. We conclude

$$f(x, y, z) = xe^{yz} + C.$$