Math 324 Quiz 5

March 7, 2017

Problem 1. For each of the following statements, write TRUE if the statement is true and FALSE if the statement is false (in big capitalized letters)! It is not necessary to show your work, and you may assume all vector fields and scalar functions are smooth. Note: not all expressions make sense.

(a) for any vector field $\vec{F}$, $\text{div}(\text{curl}(\vec{F})) = 0$

(b) for any vector field $\vec{F}$, $\text{curl}(\text{div}(\vec{F})) = \langle 0, 0, 0 \rangle$

(c) for any scalar function f, $\text{curl}(\text{grad}(f)) = \langle 0, 0, 0 \rangle$

(d) for any scalar function f, $\text{div}(\text{grad}(f)) = 0$

(e) if $\vec{F}$ is divergence-free and non-rotational, then $\vec{F} = \nabla f$ for f a solution of Laplace's equation $\Delta f = 0$.

Solution 1. True,False,True,False,True

Problem 2. For each of the following vector fields $\vec{F}$, determine whether the vector field is conservative. If it is, **bonus pts** for finding f with $\nabla f = \vec{F}$.

(a)

$$\vec{F}(x, y, z) = \langle 1 + y^2 z^3, 2y + 2xyz^3, \cos(z) + 3xy^2 z^2 \rangle.$$

(b)

$$\vec{F}(x, y, z) = \langle yz \cos(xy), xz \cos(xy), xy \sin(xy) \rangle.$$

Solution 2.

(a) Direct calculation shows that

$$\nabla \times \vec{F} = \langle 0, 0, 0 \rangle,$$

and therefore $\vec{F}$ is conservative. This implies that there exists f satisfing $\vec{F} = \nabla f$.

EXTRA CREDIT PART:
To actually find it, we can use partial integration. We have

$$f = \int f_x \partial x = \int (1 + y^2 z^3) \partial x = x + xy^2 z^3 + g(y, z).$$

Furthermore, by taking partial derivatives

$$2xyz^3 + g_y(y, z) = f_y = 2y + 2xyz^3$$

$$3xy^2 z^2 + g_z(y, z) = f_z = \cos(z) + 3xy^2 z^2$$

and therefore

$$g_y(y, z) = 2y, \quad \text{and} \quad g_z(y, z) = \cos(z).$$

It follows that

$$g(y, z) = \int g_y \partial y = \int 2y \partial y = y^2 + h(z),$$

so that

$$h'(z) = g_z(y, z) = \cos(z).$$

Therefore $h(z) = \sin(z) + C$, and it follows $g(y, z) = y^2 + \sin(z) + C$ so that

$$f(x, y, z) = x + xy^2 z^3 + y^2 + \sin(z) + C.$$

(b) We calculate the x-component of the curl of $\vec{F}$ to be

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = -x \cos(xy) \neq 0.$$

Therefore the curl itself is nonzero, and the vector field is not conservative.