Math 324 Quiz 6 Practice Solutions

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Problem 1. State Green's theorem.

Solution 1. Given a vector field $\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$, we define

$$\omega = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Then for any closed curve C with interior region R, we the counter-clockwise line integral of $\vec{F}$ is related to the area integral of ω on R by

$$\oint_C \vec{F} \cdot d\vec{s} = \int\int_R \omega dA.$$

[Note: this requires that $\vec{F}(x,y)$ is sufficiently nice (differentiable, etc) on a simply connected domain containing C]

Problem 2. Use Green's theorem to calculate the (counter-clockwise) integral of $\vec{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ over C, where C is the triangle with vertices $(0,0), (0,4)$ and $(2,0)$.

Solution 2. We calculate

$$\omega = y + \cos(x) - x\sin(x) - (\cos(x) - x\sin(x)) = y.$$

Therefore

$$\oint_C \vec{F} \cdot d\vec{s} = \int\int_R \omega dA = \int_0^2 \int_0^{4-2x} y dy dx = -\frac{1}{12}(4-2x)^3|_0^2 = \frac{16}{3}.$$

Problem 3. Find a parametric equation for the part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane.

Solution 3.
Solution Method I: If we're to the left of the xz-plane, then $y < 0$. Solving for y, this says

$$y = -\frac{1}{2}\sqrt{1 - 3z^2 - x^2}.$$

Where $-1 \leq x \leq 1$ and $-\sqrt{3} \leq z \leq \sqrt{3}$. Therefore the parametrization

$$\begin{aligned} x(s,t) &= s \\ y(s,t) &= t \\ z(s,t) &= -\frac{1}{2}\sqrt{1 - 3t^2 - s^2} \end{aligned}$$

$$-1 \leq s \leq 1 \quad -\sqrt{3} \leq t \leq \sqrt{3}$$

does the job.

Solution Method II: Ellipses are almost spheres, so we could guess that a nice parametrization may involve something in the flavor of spherical coordinates:

$$\begin{aligned} x(s,t) &= a\sin(s)\cos(t) \\ y(s,t) &= b\sin(s)\sin(t) \\ z(s,t) &= c\cos(s) \end{aligned}$$

where a, b, c are some constants which describe how squished the sides of the ellipse are in comparison to the sphere. Furthermore to get just the left side of the ellipse, we should restrict $0 \leq s \leq \pi$ and $\pi \leq t \leq 2\pi$. Note also that the maximum absolute values of x, y, and z are a, b, and c respectively. Of course, from the equation for the ellipse, the maximum value of x, y, and z are $1, 1/\sqrt{2}$, and $1/\sqrt{3}$ respectively. This determines the values of a, b, c.

$$\begin{aligned} x(s,t) &= \sin(s)\cos(t) \\ y(s,t) &= \frac{1}{\sqrt{2}}\sin(s)\sin(t) \\ z(s,t) &= \frac{1}{\sqrt{3}}\cos(s) \\ 0 \leq s \leq \pi &\quad \pi \leq t \leq 2\pi \end{aligned}$$

Problem 4. Evaluate the suface integral

$$\int\int_S x^2 z^2 dS$$

where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Solution 4. We parametrize the cone by $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$

$$x = s\cos(t), \quad y = s\sin(t), \quad z = s$$

with $1 \leq s \leq 3$ and $0 \leq t \leq 2\pi$. With this in mind, we calculate

$$\vec{r}_s \times \vec{r}_t = \langle \cos(t), \sin(t), 1\rangle \times \langle -s\sin(t), s\cos(t), 0\rangle = \langle -s\cos(t), -s\sin(t), s\rangle,$$

and therefore

$$|\vec{r}_s \times \vec{r}_t| = \sqrt{2}s.$$

Hence the surface integral becomes

$$\begin{aligned}
\int\int_S x^2 z^2 dS &= \int_1^3 \int_0^{2\pi} x(s,t)^2 z(s,t)^2 |r_s \times r_t| dtds \\
&= \int_1^3 \int_0^{2\pi} s^2 \cos^2(t) s^2 \sqrt{2} s dtds \\
&= \pi\sqrt{2} \int_1^3 s^5 ds \\
&= \pi\sqrt{2}(3^6 - 1)/6.
\end{aligned}$$