Math 324 Quiz 6

March 7, 2017

Problem 1. Explain why the line integral of $\vec{F}(x,y) = \langle 0, x\rangle$ counter-clockwise around a closed curve C is equal to the area that the curve contains.

Solution 1. Let R be the region enclosed by the curve. By Green's theorem

$$\oint_C \vec{F}(x,y) \cdot d\vec{s} = \int\int_R \omega dA$$

for

$$\omega = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - 0 = 1.$$

Therefore

$$\oint_C \vec{F}(x,y) \cdot d\vec{s} = \int\int_R 1dA = \text{area of } R.$$

Problem 2. Consider the parametric equation

$$\vec{r}(s,t) = \langle \sin(s)\cos(t), 2\sin(s)\sin(t), 3\cos(s)\rangle$$

with $0 \leq s \leq \pi/2$ and $0 \leq t \leq \pi$. Describe the surface parametrized by $\vec{r}(s,t)$ Is it a cone, paraboloid, sphere, ellipsoid, hyperboloid? Is it the whole whatever-oid, or just a part? Which part?

Solution 2. The easiest way to see this is to try to find a relationship between the variables. Note that

$$x^2 = \sin^2(s)\cos^2(t)$$

$$y^2 = 4\sin^2(s)\sin^2(t)$$

and therefore

$$x^2 + \frac{1}{4}y^2 = \sin^2(s)\cos^2(t) + \sin^2(s)\sin^2(t) = \sin^2(s).$$

Also

$$z^2 = 9\cos^2(s),$$

and therefore

$$x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = \sin^2(s) + \cos^2(s) = 1.$$

Thus $x^2 + y^2/4 + z^2/9 = 1$, which is the equation of an ellipsoid. Since $0 \leq s \leq \pi/2$ and $0 \leq t \leq \pi$, it's the portion of the ellipsoid in octants I and II.

Problem 3. Let S be the surface from the previous problem. Set up (but do not evaluate) an integral in the parameters s and t which calculates the integral of xyz over S.

Solution 3. We already have a parametrization, so we're most of the way there. We calculate

$$\vec{r}_s = \langle \cos(s)\cos(t), 2\cos(s)\sin(t), -3\sin(s) \rangle$$

$$\vec{r}_t = \langle -\sin(s)\sin(t), 2\sin(s)\cos(t), 0 \rangle$$

so that

$$\vec{r}_s \times \vec{r}_t = \langle 3\sin^2(s)\cos(t), 3\sin^2(s)\sin(t), 2\sin(s)\cos(s) \rangle.$$

Consequently

$$\begin{aligned} |\vec{r}_s \times \vec{r}_t| &= \sqrt{9\sin^4(s) + 4\sin^2(s)\cos^2(s)} \\ &= \sin(s)\sqrt{9\sin^2(s) + 4\cos^2(s)} \\ &= \sin(s)\sqrt{5\sin^2(s) + 4} \end{aligned}$$

and

$$xyz = \sin(s)\cos(t)2\sin(s)\sin(t)3\cos(s) = 6\sin^2(s)\cos(s)\sin(t)\cos(t).$$

so that

$$\int\int_S xyz dS = \int_0^{\pi/2}\int_0^{\pi} 6\sin^3(s)\cos(s)\sin(t)\cos(t)\sqrt{5\sin^2(s)+4} dtds.$$