MATH 307: Problem Set #2

Due on: April 5, 2013

Problem 1 *Book Problems*

Do problems $6.1.2-3$, $6.2.3, 6.2.6$, $6.4.2, 6.4.5$, $6.5.9, 6.5.15$, $6.3.2, 6.3.14$

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Problem 2 *Total Differential of a Linear Function*

Let M be an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ an m-vector. Define a vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ by

$$F(\vec{x}) = M \cdot \vec{x} + \vec{b}.$$

Prove using the definition of differentiability that F is differentiable at every point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and

$$dF(x_1, \ldots, x_n) = M.$$

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Problem 3 *Differentiability*

In this problem, we obtain an example of a function whose partial derivatives exist at a point, but is not differentiable at that point. Suppose

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ both exist and are equal to zero.

(b) Show that f is discontinuous at $(0,0)$.

(c) Quote a theorem proved in class to conclude that f is *not* differentiable at $(0,0)$.

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Problem 4 *Differentiability Again*

In this problem, we obtain an example of a *continuous* function whose partial derivatives exist at a point, but is not differentiable at that point. Suppose

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ both exist and are equal to zero.

(b) Show that f is continuous at $(0,0)$.

(c) Prove that f is *not* differentiable at $(0,0)$ by using the definition of differentiability.

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Problem 5 *The Chain Rule*

Recall the following theorem proved in class

Theorem 1 (Linear Approximation Theorem)**.** *Suppose*

$$F(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n))$$

is differentiable at a point $(a_1,\ldots,a_n)$*, then there exists an* $m \times n$ *matrix* $M = dF(a_1,\ldots,a_n)$ *satisfying*

$$F(a_1+h_1,\ldots,a_n+h_n) = M\cdot\vec{h} + F(a_1,\ldots a_n) + E(h_1,\ldots,h_n),$$

where $E(h_1,\ldots,h_n)$ *is a function defined on an open ball containing* $(0,\ldots,0)$ *with*

$$\lim_{(h_1,\ldots,h_n)\to(0,\ldots,0)} \frac{E(h_1,\ldots,h_n)}{\|\vec{h}\|} = (0,\ldots,0)$$

In this problem, we will use this theorem to prove the chain rule for total differentials. Suppose

$$F(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n))$$

is defined on some region R of $\mathbb{R}^n$ and differentiable at a point $(a_1,\ldots,a_n)\in R$, and

$$G(y_1,\ldots,y_m) = (g_1(y_1,\ldots,y_m),\ldots,g_\ell(y_1,\ldots,y_m))$$

is defined on a region S of $\mathbb{R}^m$ with $F(R) \subseteq S$ and differentiable at the point $F(a_1,\ldots,a_n)$. Let $M = dF(a_1,\ldots,a_n)$ and $\widetilde{M} = dG(F(a_1,\ldots,a_n))$. The previous theorem tells us that there are functions $E(h_1,\ldots,h_n)$ and $\widetilde{E}(k_1,\ldots,k_m)$ satisfying

$$F(\vec{a}+\vec{h}) = M\cdot\vec{h} + F(\vec{a}) + E(\vec{h});$$

$$G(F(\vec{a})+\vec{k}) = \widetilde{M}\cdot\vec{k} + F(\vec{a}) + \widetilde{E}(\vec{k}).$$

with $\lim_{\vec{h}\to\vec{0}} E(\vec{h})/\|\vec{h}\| = \vec{0}$ and $\lim_{\vec{k}\to\vec{0}} \widetilde{E}(\vec{k})/\|\vec{k}\| = \vec{0}$.

(a) Show that

$$G \circ F(\vec{a} + \vec{h}) - G \circ F(\vec{a}) - \widetilde{M} \cdot M \cdot \vec{h} = \widetilde{M} \cdot E(\vec{h}) + \widetilde{E}(M \cdot \vec{h} + E(\vec{h})).$$

(b) Argue based on the continuity of the map $\vec{y} \mapsto \widetilde{M} \cdot y$ that

$$\lim_{\vec{h} \to \vec{0}} \frac{\widetilde{M} \cdot E(\vec{h})}{\|\vec{h}\|} = \vec{0}.$$

(c) The *operator norm* of an $m \times n$ matrix M is the finite value $\|M\|$ defined by

$$\|M\| = \max \left\{ \frac{\|M\vec{x}\|}{\|\vec{x}\|} : \vec{x} \in \mathbb{R}^n, \vec{x} \neq 0. \right\}.$$

Use the triangle inequality to show that

$$\frac{\widetilde{E}(M \cdot \vec{h} + E(\vec{h}))}{\|\vec{h}\|} \leq \frac{E(M \cdot \vec{h} + E(\vec{h}))}{\|M \cdot \vec{h} + E(\vec{h})\|} \left(\|M\| + \frac{\|E(\vec{h})\|}{\|\vec{h}\|} \right).$$

(d) Use part (c) to argue that

$$\lim_{\vec{h} \to \vec{0}} \frac{\widetilde{E}(M \cdot \vec{h} + E(\vec{h}))}{\|\vec{h}\|} = \vec{0}.$$

(e) Conclude using the definition of differentiability that $G \circ F$ is differentiable and

$$d(G \circ F) = \widetilde{M} \cdot M = dG(F(a_1, \dots, a_n)) \cdot dF(a_1, \dots, a_n).$$

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Problem 6 *Chain Rule Problems*

For each of the following choices of F and G

(i) determine the composition $G \circ F$, and state is domain and range

(ii) determine the total differentials dG and dF

(iii) determine the total differential $d(G \circ F)$ using the chain rule

(iv) check (iii) by finding $d(G \circ F)$ using the value of $G \circ F$ found in (i)

(a) $F(t) = (1, t, t^2)$, $G(x, y, z) = x^2 z - yz^2$

(b) $F(u, v) = (v, uv, u)$, $G(r, s, t) = (\sin(rt), \cos(s))$

(c) $F(x, y, z) = (x - y, y - z, z - x)$, $G(x, y, z) = (x^2, xy, y^2)$

(d) $F(t) = (\sin(t), \cos(t))$, $G(x, y) = (u^3 + uv^2, u^2 v + v^3)$

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Problem 7 *More Chain Rule Problems*

Let $E(x, y) = (x^2 - y^2, 2xy)$, and furthermore suppose that $F(u, v)$ is some function differentiable everywhere. If $G(x, y)$ is the function defined by

$$G(x, y) = F(E(x, y)),$$

use the chain rule to determine $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ in terms of $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$.

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Problem 8 *Independent Project*

Please complete the following (to be *typed up* and turned in separately from this homework):

(a) Decide on a topic for your independent project

(b) Write a few paragraphs on what you will specifically learn and write about, why it is interesting to you, and why it should be interesting to other math folks

(c) Identify and list at least four sources of information that you can use when reseaerching your topic of choice. At least two of these should be books or journal articles (as opposed to web pages and other sources).

(d) Determine and explain at least one way that you can get your hands dirty with your topic of choice. This could involve rederiving some of the pertinent equations, applying a theory to certain specific examples of interest, or writing a computer program to help in understanding of your topic.

If you're having a hard time choosing a topic, some very nice possibilities are the following:

> Julia sets; point set topology; Navier-Stokes equations; Fourier transforms; Methods for solving nonlinear systems of equations; dynamical systems; random walks on graphs

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