Analysis for Simple Fourier Series in Sine and Co-sine Form

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Abstract

From the history of Fourier series to the application of Fourier series will be introduced and analyzed to understand mechanism of Fourier series. We've learned or probably heard already about Fourier series. However, we've learned to how to use with equation that is written in the textbook or some mathematical journals.This will guide, from beginners to experts, enhance thoughts about Fourier series.

1 Introduction

Fourier series is simply decomposing periodic functions to summation notation of simple sine and cosine to immitate the graph of periodic fucntion. It may not be perfectly same as smooth periodic functions such as multi-conditional function. However, it accordingly represent periodic function with summation notation that we've worked on often in mathematical progression.

Figure 1 is brief example of graph of Fourier series that we will later. Straight lines are multi-conditional functions that is periodic fucntion which repeats itself with certain period specified and the curved or smooth curves are Fourier series with different inputs, which will be explained later.

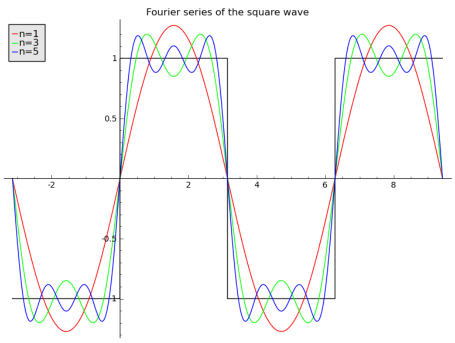

Figure 1: Graph of Fourier Series

2 History of Fourier Series

It begins as study of trigonomatric series by famous Euler and Bernoulli later Jean-Baptiste Joseph Fourier contributed most to creating Fourier series as for solving heat equation in metal plates. However, due to lack of knowledge and introduction of integrals it was not accurate and understandable. After long time, Dirichilet and Riemman completes Fourier series with great precision. The study of Fourier series is still going on these years and we call the study of Fourier series as Fourier analysis.

3 Equations

We will discuss equation of Fourier series. Fourier series is composed of several different kind of functions such as simple or general and exponential. However, we will focus on the simple Fourier series rather than advanced Fourier series.

Let $f(x)$ refers to some periodic function with the period of $T = 2L$ which means one full cycle of $f(x)$, where T refers to period of $f(x)$, would be $-L \leq x \leq L$

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nt) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L}nt) \tag{1}$$

As we see, there are some terms and variables that we need to know before we actually calculate or trasnform periodic function to Fourier series.

3.1 Terms and Variable Explaination

According to Fourier series above we need to know what does each terms and variables mean such as a_0, a_n, b_n, and L

- L refers to half the period of $f(x)$ which is periodic fucntion and which expresses domain of one full cycle of $f(x)$ is equivalent to $-L \leq x \leq L$. Often we express period of $f(x)$ as T where $T = 2L$

- a_n and b_n is defined as coefficient of Fourier series. Separately, a_n is even coefficient of $f(x)$ and b_n is odd coefficient of $f(x)$ where:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{\pi}{L}nx)dx, n \geq 0 \tag{2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{\pi}{L}nx)dx, n \leq 1 \tag{3}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{4}$$

- a_0 refers to the 0-th term of even coefficient of $f(x)$

3.2 Simple Fourier Series

According to equation (1), which is Fourier series in formation of sine and cosine terms. The simplest Fourier series we can assume is that the periodic fuction

$f(x)$ with the period of $T = 2\pi$ which means $L = \pi$, then:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Now Let $f(t)$ be equivalent to the equation of Fourier series then:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt)$$

4 Special Cases of Fourier Series

Previously, we introduced to the simple Fourier series which is periodic function $f(x)$ with the period of 2π. In this function there exists some of special cases that we can derive Fourier series into some simple and understandable functions. In this section, we will introduce you to the even and odd functions of Fourier series

4.1 The Integral of product of odd and even function

Theorem. Suppose $f_e(x)$ refers to even function and $f_o(x)$ refers to odd function, then the integral of product of $f_e(x)$ and $f_e(x)$ is equivalent to,

$$\int_{\infty}^{\infty} f_e(x)f_o(x)dx = 0$$

To prove this we expand original equation to,

$$\int_{\infty}^{0} f_e(x)f_o(x)dx + \int_{0}^{\infty} f_e(x)f_o(x)dx$$

then we subsitute $-x$ for x and then $-dx$ for dx in first term of expanded equation ,then we get,

$$\int_{\infty}^{0} -f_e(-x)f_o(-x)dx + \int_{0}^{\infty} f_e(x)f_o(x)dx$$
$$= \int_{0}^{\infty} f_e(-x)f_o(-x)dx + \int_{0}^{\infty} f_e(x)f_o(x)dx$$
$$= \int_{0}^{\infty} f_e(-x)f_o(-x) + f_e(x)f_o(x)dx$$

then now we subsitute $f_e(-x)$ for $f_e(x)$ and $-f_o(-x)$ for $f_o(x)$ then we get,

$$\int_{0}^{\infty} f_e(-x)f_o(-x) - f_e(-x)f_o(-x)dx = 0$$

therefore, we proved that the intergral of even and odd function is equal to zero ranges over $-L \leq x \leq L$

4.2 Even Function

Suppose $f(x)$ is even function, and if we use properties of cosine which is,

$$\int_{-\pi}^{\pi} \cos(x)dx = 0$$

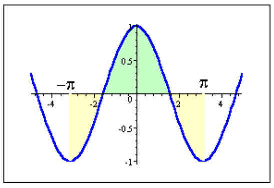

Figure 2: Graph of Co-sine function

because cosine is also a even function. According to $f(x)$ is even function $f(-x) = f(x)$ and properties of even function, we know that b_n becomes 0

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{\pi}{L}nx)dx = 0$$

because period $f(x)$ is even fucntion and ranges over $-L \leq x \leq L$, $\int_{-L}^{L} f(x)dx$ and $\sin(x)$ are odd functions by the theorem above, the integral of odd function times even function is equal to 0, therefore we can express equation of Fourier series only with a_0 and a_n due to the fact that $b_n = 0$ which,

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nt) \quad (5)$$

4.3 Odd Function

Now, suppose $f(x)$ is odd function, and if we use properties of cosine which is,

$$\int_{-\pi}^{\pi} \sin(x)dx = 0$$

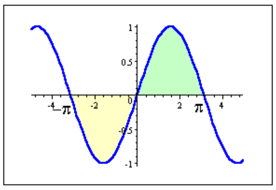

Figure 3: Graph of Sine function

because sine is also a odd function. According to $f(x)$ is odd function, $f(x) = -f(-x)$ Like Fourier series with even function of $f(x)$ we know that a_n becomes 0

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{\pi}{L}nx)dx = 0$$

because period $f(x)$ ranges over $-L \leq x \leq L$ and $f(x)$ is odd function and $cos(x)$ is even fucntion, the integral of the product of even and odd function is 0 according to theorem above, there fore $a_n = 0$ then we can express Fouriere series only with b_n

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L} nt) \tag{6}$$

5 Application of Simple Fourier series

So far, we've covered basic sine and cosine Fourier series and its special cases with variables explained. in this section, we are going to understand process of Fourier transform using actual example.

5.1 Examples of Transformation

Let $f(x)$ be a periodic function:

$$f(x) = \begin{cases} 0, & if \ -\pi \leq x < 0 \\ 1, & if \ 0 \leq x < \pi \end{cases}$$

then find Fourier coefficient and Fourier series.

First of all we need to gather some information before we transform this piecewise fucntion into Fourier series. Since $f(x)$ ranges over $(-\pi, \pi)$ we assume that period $T = 2\pi$ or $L = \pi$ which means that $f(x + 2\pi) = f(x)$.

Now, we know the period we can compute Fourier coefficient which is a_n and b_n where $L = \pi$:

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{\pi}{L} nx) dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{\pi}{L} nx) dx \\ a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) dx \end{aligned}$$

then,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{0} f(x) dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{0} 0 dx + \frac{1}{\pi} \int_{0}^{\pi} 1 dx \\ &= \frac{1}{\pi} (0 + \pi) \\ &= 1 \end{aligned}$$

Now, we solve a_n:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\frac{\pi}{\pi}nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cos(nx)dx \\
&= 0 + \frac{1}{\pi}\Big(\frac{\sin(nx)}{n}\Big|_0^\pi\Big) \\
&= \frac{1}{n\pi}(\sin(n\pi) - \sin(0)) \\
&= 0, \;\; for\ n = 1, 2, 3, \cdots
\end{aligned}$$

Now, we solve b_n

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(\frac{\pi}{\pi}nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1\sin(nx)dx \\
&= 0 - \frac{1}{\pi}\Big(\frac{\cos(nx)}{n}\Big|_0^\pi\Big) \\
&= -\frac{1}{n\pi}(\cos(n\pi) - \cos(0))
\end{aligned}$$

then,

$$b_n = \begin{cases} 0, & if\ n\ is\ even \\ \frac{2}{n\pi}, & if\ n\ is\ odd \end{cases}$$

Therefore, $a_0 = 1$, $a_n = 0$ and b_n is solved above
Let $f(t)$ be Fourier series, $f(t)$ can be written as:

$$\begin{aligned}
f(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(\frac{\pi}{L}nt) + \sum_{n=1}^{\infty} b_n\sin(\frac{\pi}{L}nt) \\
&\sim \frac{1}{2} + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} b_n\sin(nt)
\end{aligned}$$

Since we know that $b_n = 0$ when n is even, then

$$f(t) \sim \frac{1}{2}\ if\ n\ is\ even$$

If n is odd then,

$$f(t) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(nt)$$

We know that n is odd, then we can write n as $n = 2k - 1$ where $k \in Z$

$$f(t) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}\sin((2k-1)t)$$

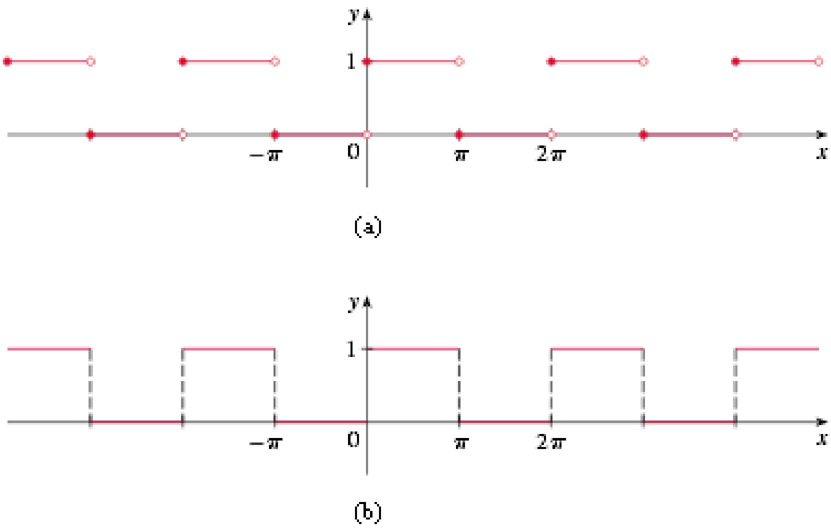

Figure 4: Graph of Square-wave Function, f(x)

5.2 Graphical Representation

Now, we will analyze graphs of example Fourier series we solved above:
In the figure 4, we see two types of graph. (a) is representation of $f(x)$ itself with open-dots and closed-dots. (b) is going to be what Fourier series will look similar.

Now we will graph $f(t)$ we calculated above which is:

$$f(t) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin((2k-1)t) \; where \; k \in Z$$

then the graph will look similar:

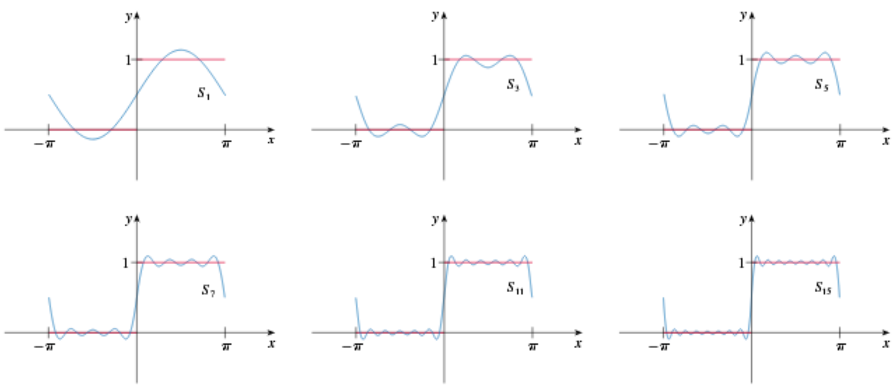

Figure 5: Graph of Fourier Series of f(x), f(t)

In the figure 5, graphs represent Fourier series as $S_k(t)$ and we see important truth about k in this kind of function and n in some other kind of functions. We see that as n increases, accuracy of approximation of Fourier series increases. For example, if you carefully S_1t it is more like sine curve which is smooth curve. However, if you take a look at the grapsh of $S_{15}t$ then it is more like composite function of vertical and horizontal lines which is similar to (b) in figure 4. It is due to properties of sine and co-sine function that is well known such as $sin(cx)$. As c increases natural period of $sin(x)$ which is 2π will decrease to π which means $sin(cx)$ has more full-cycle curve than $sin(x)$ does. In Fourier series n does same thing that c does in $sin(cs)$ vs. $sin(s)$. Therefore, while S_1t has countable wave of curves, $S_{15}t$ has multiple waves of curves to make Fourier series more accurate than that of S_1t

6 Conclusion

Fourier series interesting and some what easy-to-understand function. However, it should look like this because we just looked into simple Fourier series with sine and co-sine functions. Fourier series is more like graphical representation of mathematics rather than numerical representation. It will be much understandable when its graph is drawn on the paper rather than expanding series until we see some representation. Also, it is interesting to see that transforming some piecewise or discontinuous function to continuous function by summation notation. Eventhough it is just approximation, and as we know that as n increases, accuracy increases. If n goes to infinity, we are sure that there is noticable difference between original function and Fourier series.

6.1 For Future Studies

For future studies, i would learn about advanced Fourier series with e^k function rather than sine and co-sine function. Also, it is interesting to see that there is convergence theorem for Fourier series. I'm not sure how some periodic funtion converges to sum value while they are repeating functions, however it would be perfectly interesting to see how does that work.

7 Sources

Idea of Fourier Series:
http://calvin.enlov.net/blog/2013/02/super-early-preview-fourier-series

History and Advance Equation of Fourier Series:
http://en.wikipedia.org/wiki/Fourier_series

Information about Even and Odd Fourier Series:
http://www.enm.bris.ac.uk/admin/courses/EMa2/Lecture%20Notes%2009-10/FourierSummary.pdf

More on Even and Odd Function:
http://www.cs.unm.edu/w̃illiams/cs530/symmetry.pdf

Properties of Even and Odd Functions:
http://www.intmath.com/fourier-series/3-fourier-even-odd-functions.php

Wikipedia for Detailed Fourier Series:
http://en.wikiversity.org/wiki/Fourier_series

Graph of Sine and Cosine Function:
http://www.intmath.com/fourier-series/helpful-revision.php#prop

Examples and graphs of example is from:
http://www.stewartcalculus.com/data/CALCULUS%20Early%20Vectors/upfiles/FourierSeriesEV.pdf