**Implications of Assumptions: A Study into Modern Zermelo-Fraenkel Set Theory**

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1. **ABSTRACT**

The purpose of this short treatise is to look at a few of the assumptions which are the building blocks of Zermelo-Fraenkel Set Theory, the theory that is used to formulate and study most of conventional mathematics. Most of the important assumptions and crucial axioms will be discussed in detail, and speculation about what would occur in the case that we do not use ZFC will be considered. (For the purpose of this paper, Zermelo-Fraenkel Set Theory will often be abbreviated ZFC) By the end of the paper, the reader should be able to understand what makes the ZFC work, why there must be assumptions made, and why we use ZFC instead of other competing set theories in most of mathematics. Also, if the reader wishes to explore the topic further, this paper will attempt to show where one can go to do so.

1. **INTRODUCTION**

What is Mathematics? Numbers are the fundamental parts of many fields including finances, history, science, and even sports… they are everywhere. What makes math mathematics, though? Why is it that we have addition, subtraction, multiplication, division, etc. exists, and what properties must be fulfilled so that what is considered to be math actually works? When looking at the fundamentals of what makes math work, it can be discovered that Zerrmelo-Fraenkel Set Theory, or ZFC, is at the core. A big part of ZFC is centered on a few assumptions and the rest of ZFC is derived from these axioms and a few others. So, what are those axioms? What would occur if they weren’t followed? Would math work in other ways? These are some questions that bear looking into for those who wish to understand why math works instead of just seeing that it works.

1. **BACKGROUND**

When diving into conceptual mathematics, it is important to note that algorithms cease to become so important. This isn’t to say that we don’t use numbers or computation; instead, theories, corollaries, and other such things that define mathematics center more around ideas and symbols so that a fact can be true for all numbers and not just certain numbers. Although this seems crazy – no one would ever learn math if they had to deal with symbols at the beginning of their learning – the fact is that such an approach is actually more logic-based than just looking at specific numbers.

ZFC was first developed by Ernst Zermelo and Abraham Fraenkel, after whom the theory is aptly named along with the C, which stands for choice (the Axiom of Choice). Although argued for centuries by mathematicians and fine-tuned as more and more concepts have been discovered, the mathematics that most everyone uses today has ZFC at the base of it. At the base of that are a few assumptions, which, with a few other axioms, comprise the rest of ZFC by the process of the construction of theorems.

1. **HOW ZFC CAME TO BE**

Mathematics has existed for centuries – records of it exist as far back as Ancient Egypt. However, it was the Greeks and the Arabs that started to think about what made math work. With the Greek way of looking at the world in a logical fashion and the Arabic insistence on keeping careful records of everything, most of science came to be, and math was no exception. As the Greeks came up with the concept of the atom (something that cannot be cut), they started to come up with sets as well.

Zermelo-Fraenkel Set Theory (with the Axiom of Choice), or ZFC, relies on the fact that sets are not definable. It would be tempting to say that sets can be defined by the elements within it, but this is disputed by Russell’s Paradox, which states that if there was a set of all sets such that this set of all sets didn’t include itself, it couldn’t exist. Thus, based on the fact that sets just exist and cannot truly be defined, Zermelo and Fraenkel built their theory. Their work, combined with the Axiom of Choice (discussed later) has been called ZFC, and is the beginning blocks upon which naïve set theory is based upon.

There are other set theories (also discussed later) which are based on assuming that classes exist, and defining sets by those classes. Even more set theories do not take into account the Axiom of Choice. ZFC’s naïve set theory and these competing ideas make up what mathematicians now consider as modern set theory.

1. **HOW TO BUILD A THEORY**

In all of scientific disciplines, of which math is one (although being a field in itself), theories are created by using previously proven theories and facts to try to show why a particular phenomenon exists. A person looked at the initial problem once and tried to deduce why it was that way. After coming up with a possible reason, people spent time proving that their reason was right. Sometimes, people figured out they were wrong, but, over time, humanity was able to develop the theories of physics, chemistry, and of course, mathematics.

The golden rule in theoretical science is that nothing can be used unless it has been defined, assumed, or proven. Theories are proven using definitions and assumptions or other theories.

1. **THE ASSUMPTIONS**
2. **Assumption of Equality**

Let *A* and *B* be sets. Then *A* = *B* if and only if

$$∀ x, x\in A \leftrightarrow x \in B.$$

 What does this mean? This is the statement, “For all *x*, if *x* is in *A* and *x* is in *B*, then *A* = *B*.” Inherently, this statement is trivial as two sets *A* and *B* will be equal only when all of *x* is in all of *B*, but this can be used in some sense to prove any two sets are equal given that the elements within the sets are the same. This assumption is one of those that hold true in most all set theories, because otherwise, there is no way to prove that anything is equal. Trivial, yet absolutely necessary in order to have any sense of mathematics work.

 More importantly, this assumption is used to define addition and by addition, subtraction and the rest of the basic functions. That in turn allows the definition of natural numbers, integers, rational numbers, real numbers, and complex numbers.

 Pretty much, if there ever exists a system where the assumption of equality does not hold, then it’s hard to do anything with it as then there is no way to equate one expression to another.

1. **Assumption for Singletons**

If *a* is an entity, then there exists a set *A* such that

$$∀ x, x\in A iff x=a.$$

“For all *x*, *x* is in *A* if and only if *x* is in *a*.” As *a* denotes every element of *A*, this is pretty simple to understand. If this didn’t hold, then there would not be an easy way to talk about an element of a set without mentioning that a symbol stands for an element of a set. Not much more to state about this assumption.

 Interestingly enough, convention states that one only uses *a* to represent an element of *A*, not *b* or any other letter unless stated otherwise. Usually, *a0*, *a*1, … , *a*n is also often used.

1. **Assumption for Unions**

Let *A* be a set and let *Sa* be a well-defined set for each $a \in A.$ Then there exists a set *S* such that for all *x*,

$$x \in S \leftrightarrow ∃ a \in A \ni x \in S\_{a}.$$

This statement is known better as the $∪$ symbol, with $A ∪B$ meaning that the element is either in *A* or in *B*. This assumption is used primarily to prove different sets being equal to other sets. The symbol related to this symbol is the $∩$ symbol standing for intersection.

With unions and intersections, sets can be defined by other sets, and hence set theory can be established.

1. **Assumption for the Power Set**

If *S* is a set, then there is a set *T* for which

$$∀ x, x\in T iff x ⊆S.$$

Power sets are defined to be the set of all sets that have any or all of the elements. It’s used to talk about every possible combination of elements in a set at once. It can be counted out possibly, but it’s such a convenient tool that it is awesome to have this assumption.

1. **Assumption for the Null Set**

There exists a set *S* such that

$$∀ x, x\notin S.$$

 So this assumption is actually a very important one that distinguishes ZFC from other competing set theories. If there is no empty set, then also induction by descent cannot occur. Induction by descent is used by assuming that *P(k)* is true. Then, one shows that *P(n)* false implies *P(i)*  is false for some *i* < *n*. This shows that *P(n)*  doesn’t have a least element and therefore must be empty, so *P(n)* is true for all *n* ≥ *k*. The null set is otherwise symbolized by$ ∅$, which is actually a unique set.

1. **WHEN ZFC IS NOT FOLLOWED**

What happens if one does not follow ZFC? Although there can be a system that has some of the axioms hold true, but even if just one of the axioms fall apart, there will be things that cannot be proven true that is held for granted currently.

Let’s take each of the axioms/assumptions and see some examples of what could happen if they didn’t hold. It is true that the previous section offered token examples, but this section will seek to examine other ones in more depth.

If there a set theory existed, it would be logical to assume that one could have two things equal to each other. If *A* consisted of the set of all humans and *B* consisted of the set of all humans, then it would be logical to say that *A* is equal to *B* since every human is a human. This is where the assumption of equality comes from, and it’s imperative that this assumption hold true in every set theory. Especially in more complex cases where the sets in question (in this case *A* and *B*) are not as simple, the fact that they cannot be proven equal will translate to not being able to refer to them simply as a set *C*. This, in turn, correlates to not being able to define addition, and by that, the rest of the basic functions – subtraction, multiplication, and division.

As a set is assumed (or defined, but just not in a mathematical way) to be composed of its elements, it is useful to be able to refer to an element of a certain set without specifically stating that it’s an element of that set. The conventional method of getting this done is to use the lower case letter of the set, which is usually denoted by an upper letter. If this is not done, then it would not be assumed that *a* is an element of *A*. Instead, every time a mathematician wants to talk about an element, the said element would have to be specifically stated to be part of a specific set.

Other times, it is more useful to define a set by the combination of two other sets. This is called the union of two sets, and is defined as any element that is in either of the two (or more) sets. Suppose that the assumption for unions didn’t hold. Then it would not be possible to talk about a set that has either of the characteristics of the member sets and possibly but not necessarily all of them. If it becomes impossible to talk about a collection of sets without mentioning each of the member sets within it, it will be hard to avoid confusion. Two sets cannot be combined into another set without this union concept, and in conclusion, without the assumption for unions, $N∪Z∪R∪Q\ne C$, which messes up the understanding of the definition of sets of numbers that are used in elementary and intermediate mathematics.

One of the most used sets in all of mathematics is the power set. Actually, without the assumption for unions holding, a power set could never truly be defined, as it is known to be the collection of all sets that hold at least one of the elements of the big set, plus the null set itself (which is actually an assumption on its own). With the power of the power set (no pun intended), one can talk about all possible subsets of a set all at once, like the fact that *P*{1,2,3,4,5} are all positive numbers, excluding the null set. Otherwise, it would be necessary to list all of the combinations of these numbers to show that they are all positive. Grouping tools such as the assumption for unions and the assumption of power sets allow mathematicians to specifically talk about groups in a general manner.

There are two schools of thought in the assumption of the null set. The first is that it is indeed an assumption, and without it, one cannot prove that a set that contains nothing exists. Actually, this set in itself is kind of an oxymoron because a set containing no elements can actually exist, with the “non-element” being the only element within it. This in turn is usually used in logic-based mathematics by playing a part in a form of induction known as the method of descent, which is used by proving that a set has no least element, or that the set of least element in that particular set is actually the null set. A competing school of thought is that the assumption of the null set is not actually an assumption but could be proved to exist since there should be a possibility of not one thing being in a set, such as a set in the real numbers that fulfill the function, *x2*=-1. As the answers of this set are *i* and –*i*, and those two roots belong to the set of complex numbers, the answer set in the set of real numbers should be the null set. Whether or not the assumption is truly an assumption is not the real problem; at least the set is thought of as to exist, and can be used in math.

As shown, if one of these axioms does not hold, math as is commonly known ceases to function. It is like trying to pull out one of the foundational bricks of a building and wishing that the building will not fall. There is a reason that they are known as “axioms” or concepts that must just be defined to be true in order to build something on top of it.

1. **FURTHER INFORMATION**

If one wishes to study further on this topic, it would be prudent to look up books on other set theories and also on Zermelo and Fraenkel themselves. Otherwise, one could study more about the fundamentals of mathematical theory and about how logical theory works, as math is based on logic itself.

In fact, it is actually possible to develop another set theory on one’s own, as long as one starts with some axioms (facts that do not change) and extrapolate upon it.

World’s best universities’ professors have written books upon the topic after the formulation at the beginning of the twentieth century. As more and more mathematicians study the topic further, more material will become introduced and studied. Perhaps one could become the next person to formulate a new set theory?

1. **COMPETING SET THEORIES**

Although comprehensive of most beginning mathematics and being used by mathematicians all across the world, naïve set theory does have a few problems.

First, ZFC wouldn’t be ZFC without the C, which stands for “choice”, or Axiom of Choice. The Axiom of Choice states that if there are an infinite number of objects, there are infinite ways of arranging those objects. This is considered an axiom because of the fact that it’s impossible to prove that there are those infinite ways (due to the fact that it’s impossible to actually write down an infinite number of things). As the Axiom of Choice is used numerous times to prove different theorems (this is very useful to state every element of *A*, notably *A1, …, Ak*), it is considered to be one of the most defining things about ZFC, and is the reason why there’s the C with the ZF.

With the introduction of modern technology, most notably calculators and calculating machines (computers), a problem arose: computers couldn’t understand the concept of infinity, as no computer could truly compute infinite number of things. Hence, there are other set theories that do not involve the Axiom of Choice but instead rely on concepts and numbers that can actually be put into a computing system.

The Axiom of Choice and the fact that a set cannot truly be defined mathematically remain the biggest points of contention to mathematicians.

1. **CONCLUSION**

ZFC is the set theory that is commonly used to learn and explore the field of mathematics, and from that, the sciences and beyond. At the core of ZFC, as the nuclei upon which the rest of its concepts, corollaries, and theorems are defined, lie the basic assumptions and axioms. The assumption of equality, the assumption of singletons, the assumption for unions, the assumption of power sets, and the assumption for the null set are some of the most important assumptions without which mathematics would be more complicated to understand and harder to read.

This treatise was intended to explore those assumptions and the consequences of them being true as well as the possible outcomes and drawbacks of them being false. It also mentioned competing set theories and what makes ZFC better and more commonly used than its competitors. Hopefully, the reader will be able to have a better understanding of the fact that ZFC is inherent in the mathematics that they have been able to learn so far. If one wishes to learn more about ZFC or competing set theories, one only has to simply look for one of the many books or articles on the subject, as discussions about ZFC and its competitors are at the heart of theoretical mathematics.

As the years go on and more and more mathematicians learn about and research this topic, hopefully ZFC will become fine-tuned and therefore become an even more fundamentally sound theory for the level of importance in the role that it plays in modern society.

1. **REFERENCES**

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