# Bifurcation of Quadratic Functions

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## 1 Abstract

This is an introduction to concepts, including orbits, fixed points and bifurcation, that will help the reader's understanding of the complex nature of dynamic systems. As well as providing definitions, theorems and proofs, there is an analysis of how different parameters affect the quadratic function, $Q_c = x^2 + c$, where $c$ is a parameter. A method of graphically representing orbits is also presented to aid the reader in understanding how iterative functions behave.

## 2 Introduction

Understanding iterative functions can be useful in a range of situations, including determining the amount of money that will be in a bank account that receives 10% interest rate yearly, or predicting the growth behavior of a population. By understanding discrete dynamic systems, we can also apply that knowledge to further our comprehension of continuous dynamic systems. This paper discusses various characteristics of iterative functions and gives definitions of the relevant terms needed to understand the topic of bifurcation. Also included are theorems pertaining to fixed points and periodic points. As well as a simple procedure to create Cobweb Plots, a graphical representation of the behavior of iterative functions. The final part of this paper analyzes the affects a changing parameter has on a function. Although the concepts addressed in this paper are basic, the purpose of this paper is to exposing the reader to the topic of bifurcation and hopefully motivate him or her to pursue this topic and related topic more in depth.

## 3 Background

### 3.1 Iterations

In general, iteration is the repeating of a process. In relation to mathematics, and more specifically, functions, the process is the evaluation a function. The output of the function becomes the input of the next iteration. In order to denote which iteration we are computing, we write $F^n$, where $F$ is the function being evaluated and $n$ is the number of iterations. For example, suppose $F(x) = x^2 - 1$, then

$$F^2(x) = (x^2 - 1) - 1$$

You can iterate many functions. Some functions that are commonly iterated include, $F(x) = \lambda x(1 - x)$, $F(x) = sin(x)$ and $F(x) = \sqrt{x}$. The one that we will focus on in particular is the quadratic function, $Q_c = x^2 + c$, where $c$ is a real number. Here, the constants $c$ and $\lambda$ are called parameters.

### 3.2 Orbits

**Definition 1.1** *A seed, $x_0$, is a real or complex number used as a starting point when iterating a function $F$.*

By calculating $F(x_0)$, you will have a new number, $x_1$, which you will be able to use for your next iteration. This output will become your new input, and as before, calculate $F(x_1)$ to get $x_2$. As you generate your iterations, you will get a sequence of points, $x_0, x_1, x_2, ..., x_n$.

**Definition 1.2** *The sequence of numbers that results from iterating a function $F$ is known as the orbit of $x_0$ under $F$, where $x_0$ is the seed of the orbit.*

Here is an example, of an orbit of point 0.1 under the function $F(x) = 2x(1-x)$,

$$x_0 = 0.1$$
$$x_1 = 0.1$$
$$x_3 = 0.18$$
$$x_4 = 0.2952$$
$$x_5 = 0.41611392$$
$$x_6 = 0.48592625116$$
$$x_7 = 0.49960385919$$
$$x_8 = 0.49999968614$$
and so on...

### 3.3 Types of Orbits

As you may have guess from the recent example, an orbit can possesses a certain characteristic.

**Definition 1.3** *If a point $x_0$ satisfies the equation $F(x_0) = x_0$, where $F$ is a function, or more generally, if $F^n(x_0) = x_0$, then $x_0$ is known as a fixed point.*

**Theorem 1.4(Fixed Point Theorem)** *If $F(x)$ is a continuous function and $F(x) \in [a, b]$ for all $x \in [a, b]$ then $F$ has at least one fixed point in $[a, b]$.*

*Proof.* This proof is dependent on the consequences of the Intermediate Value Theorem, which states that if a function $F$ is continuous on $[a, b]$, and $c$ is any number between $F(a)$ and $F(b)$, including $F(a)$ and $F(b)$, then there exists a number $x$ such that $F(x) = c$. To prove the Fixed Point Theorem, suppose $F(a) = a$ or $F(b) = b$, then $F$ has a fixed point. Otherwise, $F(a) > a$ and $F(b) < b$. Let $G$ be a function such that $G(x) = F(x) - x$. Since $G(x)$ is continuous, and $G(a) > 0$ and $G(b) < 0$, there must exist a $c \in [a, b]$ such that $G(c) = 0$ and therefore a $F(c) = c$.

The orbit of a fixed point is a constant sequence $x_0, x_0, x_0, ..., x_0$. Unfortunately, the Fixed Point Theorem only comments on the existence of at least one fixed point and cannot provide an actual method to finding the point. So, in order to find the fixed points of a function $F$, find the solution or solutions to the equation $F(x) = x$. For example,

$$F(x) = 2x(1-x)$$

has fixed points when $x = 0$ and 0.5.

A fixed point can be attracting, repelling or neutral.

**Definition 1.5** *Suppose $x_0$ is a fixed point of the function $F$, then*

1. *$x_0$ is an attracting fixed point of $F$, if $|F'(x_0)| < 1$*
2. *$x_o$ is a repelling fixed point of $F$, if $|F'(x_0)| > 1$*
3. *$x_o$ is a neutral fixed point of $F$, if $|F'(x_0)| = 1$*

Attractive and repellent fixed points are sometimes know as sinks and sources, respectively. Fixed points that are neutral can behave in various ways. They can be weakly attracting, weakly repelling or neither attracting nor repelling.

**Theorem 1.6** *If $x_0$ is an attracting fixed point for a differentiable function $F$, then there is an $\varepsilon > 0$ such that $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ satisfies the following condition:*

1. $f^n(x) \in (x_0 - \varepsilon, x_0 + \varepsilon)$ *for all* $n > 0$

2. *As* $n$ *approaches infinity,* $f^n(x)$ *approaches* $x_0$ *for all* $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

*Proof.* Since F is differentiable everywhere, it is continuous. Using the Mean Value Theorem and the definition of an attracting fixed point, $\left|F^{'}(x_0)\right| < \lambda < 1$, for some $\lambda > 0$. By the continuity of the derivative at $x_0$, there is a $\varepsilon > 0$ such that $F^{'}(x_0) < \lambda$ on the interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$. So,

$$F^{'}(c) = \tfrac{F(x) - F(x_0)}{x - x_0}$$

for some $c$ between $x_0$ and $x$.
So,

$$\tfrac{|F(x) - F(x_0)|}{|x - x_0|} = \left|F^{'}(c)\right| < \lambda$$

$$|F(x) - F(x_0)| < \lambda \, |x - x_0|$$

The distance from $F(x)$ to $x_0$ decreases by a factor of $\lambda$ because,

$$|F(x) - x_0| = |F(x) - F(x_0)| < \lambda \, |x - x_0|$$

Since, $F$ maps $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ to itself. The distance from $F(x)$ to $x_0$ decreases.
It can also be said that,

$$\left|F^2(x) - x_0\right| = \left|F^2(x) - F^2(x_0)\right| < \lambda \, |F(x) - x_0| < \lambda^2 \, |x - x_0|$$

and,

$$|F^n(x) - (x_0)| < \lambda^n \, |x - x_0|$$

for $\lambda < 1$, $\lambda^n$ converges to 0. So, as $n$ approaches infinity, $F^n(x)$ approaches $x_0$.

**Theorem 1.7** *Let* $x_0$ *be a repelling fixed point of the differential function* $F$*. Then there exists an* $\varepsilon > 0$ *such that if* $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ *and* $x_0 \neq x$*, then there exists a* $n > 0$ *such that* $f^n(x) \notin (x_0 - \varepsilon, x_0 + \varepsilon)$.

*Proof.* Since $F$ is a differentiable everywhere, it is continuous. By the Mean Value Theorem and the definition of a repelling fixed point, $\left|F^{'}(x_0)\right| > \lambda > 1$, for some $\lambda > 0$. By the continuity of the derivative at $x_0$, there is a $\varepsilon > 0$ such that $F^{'}(x_0) > \lambda$ on the interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$. So,

$$F^{'}(d) = \frac{F(x) - F(x_0)}{x - x_0}$$

for some $d$ between $x_0$ and $x$.
So,

$$\tfrac{|F(x) - F(x_0)|}{|x - x_0|} = \left|F^{'}(d)\right| > \lambda$$

$$|F(x) - F(x_0)| > \lambda \, |x - x_0|$$

So, the distance $|F(x) - F(x_0)|$ can be written as follows,

$$|F(x) - x_0| = |F(x) - F(x_0)| > \lambda \, |x - x_0|$$

Since $\lambda > 1$, we have that $F(x)$ get further and further away from $x_0$. If $F(x) \notin (x_0 + \varepsilon, x_0 + \varepsilon)$, then the proof is done. But if this not the case, then repeat the process above, replacing $x$ with $F(x)$ in $(x_0 + \varepsilon, x_0 + \varepsilon)$. This results in,

$$\left|F^2(x) - x_0\right| = \left|F^2(x) - F^2(x_0)\right| > \lambda \left|F(x) - x_0\right| > \lambda^2 \left|x - x_0\right|$$

Recall, that $\lambda > 1$. So, $\lambda^2 > \lambda$.

By continuing to apply $F$ to $x$ and $x_0$, the distance between $F^n(x)$ and $x_0$ increases. In general, we have the following, under the condition that $f^{n-1}(x) \in (x_0 + \varepsilon, x_0 - \varepsilon)$

$$\left|F^n(x) - x_0\right| > \lambda^n \left|x - x_0\right|$$

So, as $n$ approaches infinity, $\lambda^n$ approaches infinity. Nearby points eventually move away from the fixed point. There exists some $n > 0$ such that

$$\left|F^n(x) - x_0\right| > \lambda^n \left|x - x_0\right| > \varepsilon$$

Now that we have discussed fixed points in depth, let us move on to another type of point. The second type of point that we will talk about is called a period point.

**Definition 1.8** *A point $x_0$ of function $F$ is periodic, if there exists an $n$ so that $x_0$ is a fixed point of $F^n(x)$, meaning it satisfies $F^n(x_0) = x_0$.*

**Definition 1.9** *A cycle of period $n$ is an orbit that repeats itself every $n$ iterations.*

For example, 0 lies on a cycle of period 2 for $x^2 - 1$ since its orbit is $0, -1, 0, -1, ...$.Furthermore, a period-2 cycle would two points, $x_0$ and $x_1$ such that $F(x_0) = x_1$ and $F(x_1) = x_0$. More generally, an n-cycle would take the form

$$x_0, x_1, x_2, ..., x_{n-1}, x_0, x_1, ...$$

To determine if a period point is attracting, repelling or neutral, recall the Chain Rule and consider the derivative $F^n(x_0)$. With the Chain Rule, we know that if we have two differentiable functions $F$ and $G$, then

$$(F \circ G)^{'}(x) = F^{'}(G(x)) \cdot G^{'}(x)$$

So,

$$(F^2)^{'}(x_0) = F^{'}(F(x_1)) \cdot F^{'}(x_0) = F^{'}(x_1) \cdot F^{'}(x_0)$$

and

$$(F^3)^{'}(x_0) = F^{'}(F^2(x_0)) \cdot (F^2)^{'}(x_0) = F^{'}(x_2) \cdot F^{'}(x_1) \cdot F^{'}(x_0)$$

So, $(F^n)^i = F^{'}(x_{n-1}) \cdot ... \cdot F^{'}(x_0)$, where $x_0, ..., x_{n-1}$ lie on a cycle of period $n$ for $F$ and $x_i = F^i(x)$. The derivative of $F^n$ at $x_0$ is simply the product of the derivative for $F$ at all points on the orbit. This means that we don't have to know the equation for $F^n$, just the points on the orbit.

## 3.4 Graphical Representation of Orbits

One tool to help quickly visualize an orbit of a dynamical system is called a Cobweb Plot. To construct a Cobweb Plot of a function $F(x)$, first draw the graphs $x = y$ and $f(x) = y$ on the same set of x and y axis. Then, plug the initial value, $x_0$ into the function $F$. On the graph, draw a vertical line from $x_0$ on the x-axis to the point $(x_0, F(x_0)) = (x_0, x_1)$. Now draw a horizontal line over the $x = y$ line. This is the point $(x_1, x_1)$. Now repeat the process. So, draw a vertical line up to $(x_1, F(x_1) = (x_1, x_2)$. Then draw a horizontal line to $(x_2, x_2)$. To create more iteration, keep repeating this process. Here is an example of Cobweb Plot:

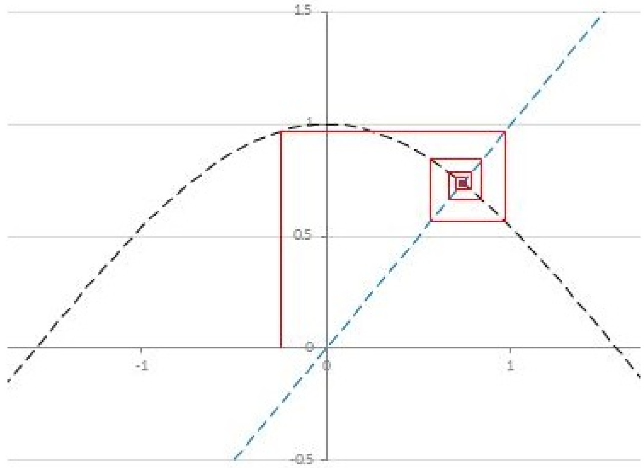


Figure 1: $F(x) = cos(x)$ and $x_0 = -0.25$

Cobweb Plots make it easy to identify the different types of points, whether they be fixed or periodic; or attractive, repellent, or neutral.

# 4 Bifurcation

Bifurcation occurs in one-parameter family of functions when there is a change in the point structure as the parameter $\lambda$ passes through some particular parameter value, $\lambda_0$.

Bifurcation refers to the changes in the set of fixed or periodic points or other sets of dynamic interest. Although there are many types of bifurcation, we will discuss two of them. The first is a Saddle-node or Tangent Bifurcation. We will see a specific example of this in a quadratic function. But for now, consider one-parameter family of functions, $F_\lambda$,with $\lambda$ being the parameter. For example, the functions $F_\lambda(x) = \lambda x(1-x)$ and $S_\lambda(x) = \lambda sin(x)$ are both one-parameter functions.

## 4.1 Saddle-node or Tangent Bifurcation

**Definition 1.10** *A function $F_\lambda$ undergoes a Saddle-node (or Tangent) Bifurcation at the parameter $\lambda_0$ if there is an $\varepsilon > 0$ such that on the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, there exists a $\lambda$ and*

1. *For $\lambda_0 - \varepsilon < \lambda < \lambda_0$, $F_\lambda$ has no fixed points in the interval*
2. *For $\lambda = \lambda_0$, $F_\lambda$ has one fixed point in $I$ and that point is neutral*
3. *$\lambda_0 < \lambda < \lambda_0 + \varepsilon$, $F_\lambda$ has two fixed points on $I$ and one attracting and another repelling*

Note, a bifurcation can also occur in the reverse direction. So, the above definition can also be written as:

1. *For $\lambda_0 + \varepsilon > \lambda > \lambda_0$, $F_\lambda$ has no fixed points in the interval*
2. *For $\lambda = \lambda_0$, $F_\lambda$ has one fixed point in $I$ and that point is neutral*
3. *$\lambda_0 > \lambda > \lambda_0 - \varepsilon$, $F_\lambda$ has two fixed points on $I$ and one attracting and another repelling*

Basically means a Saddle-node Bifurcation occurs if the functions $F_\lambda$ has no fixed points on an interval $(\lambda_0-\varepsilon, \lambda_0+\varepsilon)$ for $\lambda$-values slightly less than $\lambda_0$, one fixed point in $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ when $\lambda = \lambda_0$ and exactly two fixed points when $\lambda > \lambda_0$. Periodic points may undergo a Saddle-node Bifurcation. These are described by simply replacing $F_\lambda$ with $F^n_(\lambda)$ for the cycle of period $n$ in the above definition. Saddle-node Bifurcation typically occurs when the graph of $F_{\lambda_0}$ is tangent with the diagonal at $(x_0, x_0)$, $F^{'}(x_0) = 1$, but $F^{''}(x_0) \neq 0$. This indicates that the graph of $F_{\lambda_0}$ is either concave up or down, so that near $x_0$, $F_{\lambda_0}$ has only one fixed point $x_0$. The fact that $F_{\lambda_0}$ is tangent to the diagonal at $x_0$ is the reason for the terminology tangent bifurcation. Bifurcation theory is a local theory in that

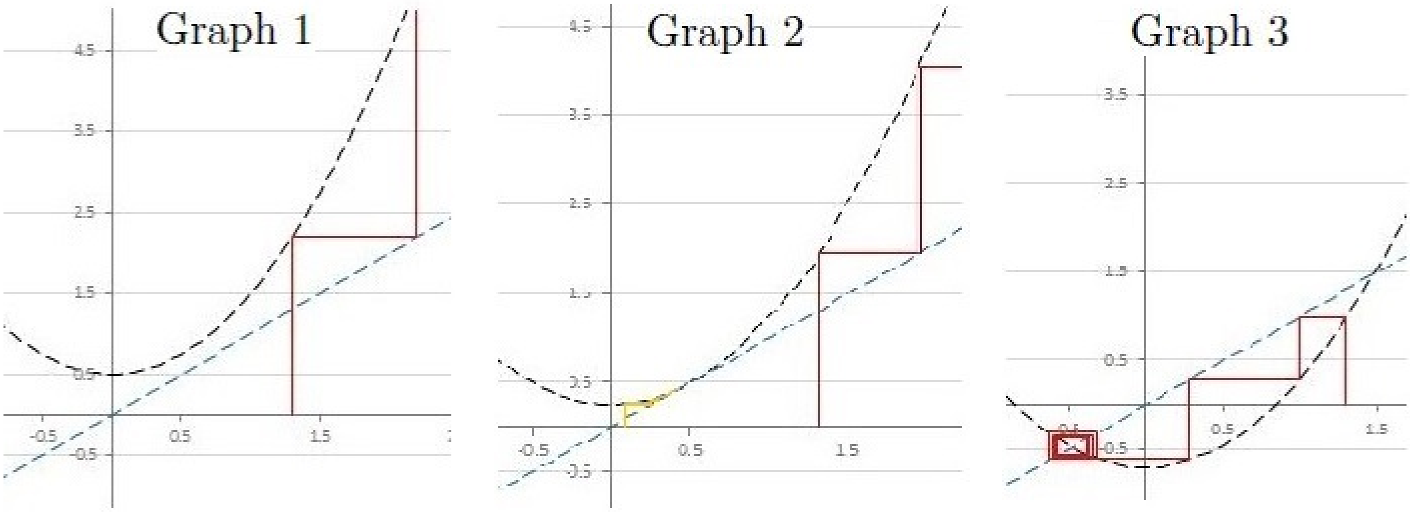


Figure 2: In Graph 1, $\lambda > \lambda_0$; in Graph 2, $\lambda = \lambda_0$; and in Graph 3, $\lambda < \lambda_0$

we are only concerned about the changes in the periodic point structure near the parameter value $\lambda_0$. That is the reason for the $\varepsilon$ in the definition ($\varepsilon$ is usually small).

An example of a Saddle-node Bifurcation, is at the parameter $\lambda = \frac{1}{4}$ of the function $Q_\lambda = x^2 + \lambda$. Let us take the interval $I$ to be the entire real line. Also, let $\lambda_0 = \frac{1}{4}$. Then conditions in Definition 1.10 are satisfied. In Graph 1, there are no fixed points when $\lambda = \frac{1}{2} > \frac{1}{4}$. In Graph 2, there is a neutral fixed point when $\lambda = \frac{1}{4}$. And in Graph 3, there is a pair of fixed points (1 attracting and 1 repelling) for $\lambda = -\frac{3}{4} < \frac{1}{4}$.

The images above, give a visual representation of a Saddle-node Bifurcation. Graph 2 also is able to show the behavior of the neutral point which acts like neither a repelling nor an attracting fixed point, depending if $x_0$ is greater than or less than 0.5 .

## 4.2 Period-doubling Bifurcation

Now that we have discussed Saddle-node Bifurcation more generally, let us do the same with the second type of bifurcation, Period-doubling Bifurcation.

**Definition 1.11** *A one-parameter family of function $F_\lambda$ undergoes a Period-doubling Bifurcation at the parameter value $\lambda = \lambda_0$ if there an $\varepsilon > 0$ such that on the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ there exists a $\lambda$ and :*

1. *For each $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, there is a unique fixed point $p_\lambda$ for $F_\lambda$ in the interval*

2. *For $\lambda_0 - \varepsilon < \lambda \leq \lambda_0$: $F_\lambda$ has no cycles of period 2 in $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ and $p_\lambda$ is attracting or repelling*

3. *$\lambda_0 < \lambda < \lambda_0 + \varepsilon_1$, $F_\lambda$: there is a unique 2-cycle $q_\lambda^1$, $q_\lambda^2$ in $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ with $F_\lambda(q_\lambda^1) = q_\lambda^2$. This 2-cycle is attracting or repelling, while the fixed point $p_\lambda$ is attracting or repelling*

4. *As $\lambda$ approaches $\lambda_0$, $q_\lambda^1$ approaches $p_{\lambda_0}$*

Thus there are two typical cases for a Period-doubling Bifurcation. As the parameter changes, a fixed point may change from attracting to repelling and simultaneously, create an attracting 2-cycle. Alternatively, the fixed point may change from repelling to attracting and create a repelling cycle of period 2. Also, cycles may undergo a Period-doubling Bifurcation. In this case, a cycle of period $n$ will give birth to a cycle of period $2n$.

Period-double Bifurcation occurs when the graph of $F_\lambda$ is perpendicular to $x = y$, or equivalently, when $F'(p_{\lambda_0}) = -1$. By the Chain Rule, it follows, that

$$(F_\lambda^2)'(p_{\lambda_0}) = F_\lambda'(F(p_{\lambda_0})) \cdot F_\lambda'(p_{\lambda_0})$$

Since $p_{\lambda_0}$ is a fixed point,

$$F(p_{\lambda_0}) = p_{\lambda_0}$$

and since

$$F'(p_{\lambda_0}) = -1$$

So,

$$(F_\lambda^2)'(p_{\lambda_0}) = -1 \cdot -1 = 1$$

The graph of the second iterate of $F_\lambda$ is tangent to the diagonal when the Period-doubling Bifurcation occurs. We will soon see an example of this when $c = -\frac{3}{4}$ for the quadratic family $Q_c(x) = (x^2 + c)$.

# 5 Quadratic Family of Functions

Now that we have gone over basic concepts and terms, we will study and observe orbits for the quadratic family of functions which includes all functions in the form $F(x) = x^2 + c$, where $c$ is any real constant. Let us denote a function in the quadratic family as $Q_c$. We will try and understand how the dynamics of $Q_c$ will change as $c$ varies. By using the method previously state, we set $Q_c(x) = x^2 + c$ equal to $x$. The two roots that we get are,

$$p_+ = \tfrac{1}{2}(1 + \sqrt{1-4c})$$
$$p_- = \tfrac{1}{2}(1 - \sqrt{1-4c})$$

So, in order for $p_+$, and $p_-$ to be real roots, $1 > 4c$, or equivalently, $c \leq \frac{1}{4}$. Let us look more closely at points $p_+$ and $p_-$, when $c < \frac{1}{4}$.

## 5.1 When $c < \frac{1}{4}$

Since $Q_c' = 2x$,

$$Q_c'(p_+) = 2 \cdot \tfrac{1}{2}(1 + \sqrt{1-4c}) = 1 + \sqrt{1-4c}$$
$$Q_c'(p_-) = 2 \cdot \tfrac{1}{2}(1 - \sqrt{1-4c}) = 1 - \sqrt{1-4c}$$

If $c < \frac{1}{4}$, as a result, $Q_c'(p_+) > 1$. This means that $p_+$ is a repelling fixed point. If $c = \frac{1}{4}$, then $Q_c'(p_+) = 1$ and so there is a neutral fixed point. Now let us look at $p_-$. If $c = \frac{1}{4}$, then $Q_c'(p_-) = 1$, which means $p_+ = p_- = \frac{1}{2}$ is a neutral point. Since if $\left|Q_c'(p-)\right| < 1$, then $p_-$ is attracting,

$$\left|Q_c'(p-)\right| < 1$$
$$-1 < Q_c'(p-) < 1$$
$$-1 < 1 - \sqrt{1-4c} < 1$$
$$2 > \sqrt{1-4c} > 0$$
$$4 > 1 - 4c > 0$$
$$3 > -4c > -1$$
$$-\tfrac{3}{4} < c < -\tfrac{1}{4}$$

Hence, $p_-$ is attracting when $-\frac{3}{4} < c < \frac{1}{4}$, neutral when $c = \frac{1}{4}$ and repelling when $c < -\frac{3}{4}$. So, as $c$ decreases from $\frac{1}{4}$ to $-\frac{3}{4}$, $Q_c$ has one single fixed point to two fixed points. This is a kind of bifurcation known as the Saddle-node or Tangent Bifurcation.

## 5.2 When $-\frac{3}{4} < c < \frac{1}{4}$

For any $c < -\frac{3}{4}$, a cycle of period 2 appears. To see this, we solve for $Q_c^2 = x$. So,

$$Q_c^2 = (x^2 + c)^2 + c = x^4 + 2cx^2 + c^2 + c = x$$
$$x^4 + 2cx^2 + c^2 + c - x = 0$$

We already know two solutions to this equation (they are the two fixed points, $p_+$ and $p_-$). Hence, $x - p_+$ and $x - p_-$ are factors for this equation. Since, $p_+$ and $p_-$ are fixed points, they solve the equation

$$x^2 + c = x$$

So,

$$(x - p_+)(x - p_-) = x^2 + c - x = 0$$

To find the other two solutions,

$$\frac{x^4+2cx^2+c^2+c-x}{x^2+c-x} = x^2 + c - x$$

Therefore, the other solutions for this quadratic are the roots, $q\pm = \frac{1}{2}(-1 \pm \sqrt{-4c-3})$. Notes, that $q$ is only real if $-4c - 3 \geq 0$, or equivalently, $c \leq -\frac{3}{4}$. As $c$ decreases below $-\frac{3}{4}$, the fixed $p_-$ changes from attracting to repelling and a new 2 period cycle appears at $q\pm$. As previously discussed, this is called Periodic-doubling Bifurcation. Notice, that when $c = -\frac{3}{4}$, we have $q_+ = q_- = -\frac{1}{2}$ which is equal to $p$. So, these two periodic points originated at $p$.

### 5.3 When $c = -2$

When $c = -2$, $p_+ = \frac{1}{2}(1 + \sqrt{1 - 4 \cdot (-2)}) = 2$. If we look at the interval $[-2, 2]$, which we will call $I$, the function $Q_{-2} = x^2 - 2$ decreases on the interval $[-2, 0]$ and increases on the interval $[0, 2]$. It is also evident, that $Q_{-2}$ maps exactly two points in $I$ to the same point. So, every point(except for $-2$) in the image of the interval $I$ has exactly 2 preimages. Also, there are two fixed points, $p_-$ and $p_+$

Now, if you look at $Q^2_{-2}$ on the interval $I$, there are 4 preimages for every point (except for $-2$). Also, the line x=y intersects with $Q^2_{-2}$ 4 times, so there are 4 fixed points.

A similar result occurs when looking at $Q^3_{-2}$ and larger iterations of $Q_{-2}$. In general, the function $Q_{-2}$ has at least $2^n$ periodic points of period $n$ in the interval $[-2, 2]$.

This seems to contradict what we saw when $c \geq -\frac{3}{4}$, which was that there are very few fixed and periodic points. But, as $c$ made a small decrease to $-2$, the result was infinitely many periodic point. As a result, the orbits have become more random. Below is a image of two orbits on the function $F(x) = x^2 - 2.2$

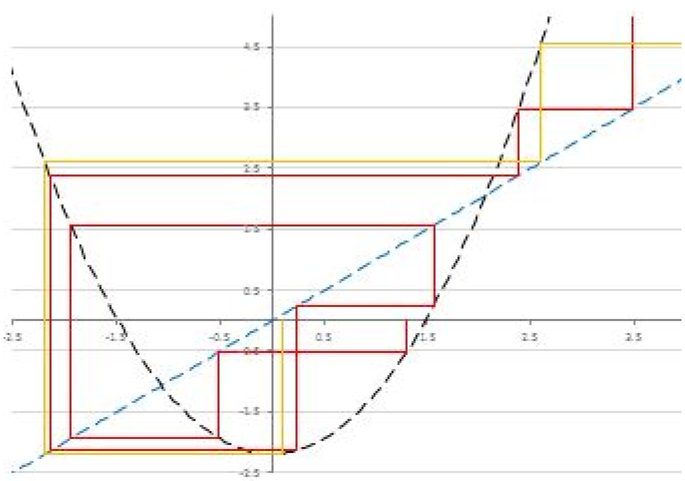

Figure 3: Orbit with inital value 0.1 (orange line) and 1.3 (red line)

## 6 Future Direction of Study

We have just briefly seen how a function can transform from simple to very complex dynamics. To learn more about the topic and gain a greater understanding of why this occurs, we would need to learn how to read orbit diagrams and use symbolic dynamics to convert the complicated behavior that we have witnessed into a different dynamical system which we can understand fully. Also, understanding Cantor's Middle-Third Set would also be useful. To find more information about this, a helpful resource would be the textbook *An Introduction to Symbolic*

*Dynamics and Coding* by Douglas Lind and Brian Marcus, and *A First Course in Chaotic Dynamical Systems: Theory and Experiment* by Robert Devaney. Learning about the properties of chaotic systems, will eventually lead to bigger and more complex topics like Sarkovskii's Theorem, Newton's Method and Fractals.

## 7 Conclusion

Hopefully after reading this paper in its entirety, you now know how to iterate functions, find fixed points or periodic points, and determine if a Saddle-point or Period-doubling bifurcation has occurred. Understanding under what conditions an repelling or attracting fixed points can occur and how, will help you get a better idea of how the dynamic system behaves as a whole. All things mentioned in this paper are to help you understand and describe dynamic system. The difficulties that are presented, for instance, like trying to find points associated with a cycle of period $n$, where $n$ is large is hopefully a motivator to discover and learn different method what would more easily finding these points. It is by thoroughly grasping these concepts will you be able to go into the complexities of this topic.

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