On The Probabilities And Characteristics Of Random Walks On Finite Graphs

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Abstract

For a given stochastic process exhibiting the Markovian property that defines a walk on a finite graph $\mathcal{G}$ defining some polyhedron or arbitary, the probability for a particle to start and end at two not necessarily distinct points after N steps is determined by the adjacency matrix $Adj(\mathcal{G})$. With this, concepts of null and non-null structures, and step saturation are to be explored.

Introduction

Combining the mathematical technologies developed in the fields of stochastic processes and graph theory, for a given finite graph that is either arbitrary or defining a polyhedron, we will determine the probability that a particle will end at some point of interest if it starts at some initial point and its walk is Markovian and random. It will be assumed that the particle may not land on the edges connecting points–if it helps with visualizing, the edges can be thought of as two way ski lifts running off a perpetual motion motor (and hence will never stop!). It will be instructive and formative to give background information about several topics that will not necessarily be used in the development of the theory presented in this paper. The reasoning for this is to build an appropriate intuition and give some background material to fall back on lest confusion arises.

Background

To begin our studies in background information we will start with some theory of stochastic processes. This will allow us to have the background to dive into Markov chains, the Chapman-Kolmogorov equation, and random walks. However, we will explore a bit of graph thoery before random walks.

Stochastic Processes[a]

We shall introduce a mathematical object called a random experiment with defining property that it can produce exactly one outcome indeterminate prior to the production. Denote the random experiment as $\mathcal{E}$ and its outcome, $\mathcal{O}(\mathcal{E})$. Each $\mathcal{E}$ will have a set of possible outcomes attributed to it. This set is called the statespace, $\mathcal{S}$. The subsets $E \subseteq \mathcal{S}$ are called events and are said to occur in $\mathcal{E}$ if $\mathcal{O}(\mathcal{E}) \in E$. Given that each $E \subsetneq \mathcal{S}$ cannot be said to definitely occur by the nature of $\mathcal{E}$, there shall be attributed to each E a probability $P(E)$ that abide the following axioms

1. $0 \leq P(E) \leq 1$
2. $P(\mathcal{S}) = 1$
3. For any sequence of events E_1,E_2,... that are mutually exclusive, that is, events for which $E_i \cup E_j = \emptyset$ when $i \neq j$, $P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$.

Given a random experiment with statespace S, a random variable X maps from each outcome in S a real value with probability of X taking a real value in a set of real values A being

$$P(X \in A) = P(X^{-1}(A))$$

where $X^{-1}(A)$ is the event consisting of all points $s \in S$ such that $X(s) = A$.
If we have a index set N and random variables $X(n)$ with $n \in N$, the collection $\underline{X} = \{X(n)|n \in N\}$ is called a stochastic process–note we can relable for convenience $X(n)$ as X_n.

Definition (Markov Chains)[a]. *Suppose we are given a stochastic process $\underline{X} = \{X_n|n \in \mathbb{N}\}$ that takes on finite values. Suppose there are fixed probabilities from $\underline{X}$ starting in state i and continues to state j and denote them as*

$$P_{ij} = P\{X_{n+1} = j|X_n = i, X_{n+1} = i_{n-1}, ..., X_0 = i_o\}.$$

The process of $\underline{X}$ transitioning from i to j is then called a Markov chain.
By the condition P_{ij} being that $X_{n+1} = j|X_n = i, X_{n+1} = i_{n-1}, ..., X_0 = i_o$ it is to be understood that the probability is attributed only to going from i to j regardless of previous states; this independence is called the Markovian property.
With this definition given, we may go on to a more general notion where we consider going from a state i to a state j after n transitions, or steps. However, the following equation is given only to mention in the **Future Directions Of Study** section and not for the theory of the subject detailed in this paper.

Definition (Chapman-Kolmogorov Equations)[a]. *Suppose we are given a stochastic process $\underline{X} = \{X_n|n \in \mathbb{N}\}$ that takes on finite values. Suppose we are interested in the probability of $\underline{X}$ starting in state i and after n steps ending in state j. We denote this as*

$$P_{ij}^n = P\{X_{n+m} = j|X_m = i\} \text{ with } n \geq 0, i, j, \geq 0.$$

To determine this probability we can use the Chapman-Kolmogorov equation

$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m.$$

What the Chapman-Kolmogorov equation is stating is that the probability from going from state i to state j in $n+m$ steps is equal to the sum of product of probabilities of transitions from state i to state k in n steps and transitions from state k to state j in m steps. To further clarify, let us go through three examples of $n + m = 2$

Example I[b]

$$P_{ij}^2 = P(X_{n+2} = j|X_n = i)$$
$$= \sum_{k=0}^{\infty} P(X_{n+2} = j, X_{n+1} = k|X_n = i)$$
$$= \sum_{k=0}^{\infty} P(X_{n+2} = j|X_{n+1} = k, X_m = i) \cdot P(X_{n+1} = k|X_n = i)$$

$$= \sum_{k=0}^{\infty} P_{kj} P_{ik}.$$

Example II

Let $n + m = 4$ in the Chapman-Kolmogorov equation. Then we see

$$P_{ij}^4 = P(X_{n+4} = j | X_n = i)$$

$$= \sum_{k=0}^{\infty} P_{kj}^2 P_{ik}^2$$

$$= \sum_{k=0}^{\infty} \left[\left(\sum_{l=0}^{\infty} P_{lj} P_{kl} \right) \cdot \left(\sum_{m=0}^{\infty} P_{mk} P_{im} \right) \right].$$

We can generalize this result further.

Example III

Or in general for $n + m = N$, if we continue to divide as we did for $n + m = 4$

$$P_{ij}^N = \sum_{k_0} \left[\left(\sum \backslash \prod (i, k_0) \right) \cdot \left(\sum \backslash \prod (k_0, j) \right) \right]$$

where

$$\sum \backslash \prod (i, k_0) = \sum_{k_1} \left[\cdots \left[\left(\sum_{k_i} P_{ik_i}^{\frac{N}{2i}} P_{k_i k_{i-1}}^{\frac{N}{2i}} \right) \cdot \left(\sum_{k_i} P_{k_{i-1} k_i}^{\frac{N}{2i}} P_{k_i k_0}^{\frac{N}{2i}} \right) \right] \cdots \right],$$

$$\sum \backslash \prod (k_0, j) = \sum_{k_1} \left[\cdots \left[\left(\sum_{k_i} P_{k_0 k_i}^{\frac{N}{2i}} P_{k_i k_{i-1}}^{\frac{N}{2i}} \right) \cdot \left(\sum_{k_i} P_{k_{i-1} k_i}^{\frac{N}{2i}} P_{k_i j}^{\frac{N}{2i}} \right) \right] \cdots \right].$$

A similar equation can be derived when n and m do not add up to an even number; to do so, just divide P^N into $P^{N-1}P^1$, and then divide the P^{N-1} as above since $N - 1$ will be even.

Graph Theory[c]

Since we will be dealing with walks on finite graphs, definitions of both need to be established.

Definition (Graph). Consider a non-empty set of points and another set containing pairs of distinct points where the points are from the former set. We say $\mathcal{G}$ is the former set with the latter assigned to it with the latter being called the lines (or edges) of the set.

Example. Consider a octahedron. Then the vertices of this polyhedron make up the set of points and the edges the set of distinct pairs of a graph (i.e., the lines).

Definition (Walk). Consider a graph $\mathcal{G} = \{g_1, g_2, ..., g_M\}$. Then a sequence of N points is a N-step walk on the graph $\mathcal{G}$ if the points of the sequence are points of $\mathcal{G}$ and are connected in the sense that there exists a line as a pair of the two points contained in the graph. A random walk will be one where the sequenced is defined by a random variable such that after a known initial point, the next point will not be known until the next step is taken.

Random Walks[d]

It will be crucial to have an example to fall back onto when trying to understand parts of this paper. This is why a random walk on a single axis with equal probability of $\frac{1}{2}$ of going one way or the other is to be explained here–with this knowledge, when trying to understand why a particle starting at a point can, via randomly walking, can arive at another, one will have a concrete example in accepting such a result.

Consider a particle initially at the origin of an axis we will call the x-axis; i.e., the particle will be at $x = 0$ at $t = 0$, where t is time. Letting Δt be the interval of time between each displacement of Δx or $-\Delta x$, we have the random variable X_i taking on $X_i = +\Delta x$ or $X_i = -\Delta x$. Since we are giving both displacements an equal probability we have $P^1_{ij} = \frac{1}{2}$ for all i and $j = i + 1$. We also have a notion of net displacement being

$$X = X_1 + X_2 + \cdots + X_n$$

for n steps. When considering the expectation value of X, $\langle X \rangle$, and variation (or standard deviation)[1] of X, $var\{X\}$, we find something interesting. For the former, from

$$\langle X_i \rangle = \frac{1}{2}\Delta x - \frac{1}{2}\Delta x,$$

we have

$$\begin{aligned}\langle X \rangle &= \langle X_1 \rangle + \langle X_2 \rangle + \cdots \langle X_n \rangle \\ &= 0.\end{aligned}$$

However, for the latter, from

$$\begin{aligned}var\{X_i\} &= \left\langle X_i^2 \right\rangle - \langle X_i \rangle^2 \\ &= \left\langle X_i^2 \right\rangle \\ &= \frac{1}{2}(\Delta x)^2 + \frac{1}{2}(-\Delta x)^2 \\ &= \Delta x^2,\end{aligned}$$

we see

$$\begin{aligned}\left\langle X^2 \right\rangle &= \left\langle X_1^2 \right\rangle + \left\langle X_2^2 \right\rangle + \cdots \left\langle X_n^2 \right\rangle \\ &= \Delta x^2 + \Delta x^2 + \cdots \Delta x^2 \\ &= n\Delta x^2 \\ &= t\frac{\Delta x^2}{\Delta t},\end{aligned}$$

[1] It is assumed the reader has came across expectation values and standard deviation by now. If not, refer to **[d]** in bibliography where this example is modeled off of.

where the last equality comes from the fact that $n = \frac{number\ of\ steps}{time\ interval}$. This result tells us something interesting given that after n steps, the average displacement from the mean is $n\Delta x^2$ and hence we would expect a particle in such a system to be found, after n steps, not at the origin.

The Adjacency Matrix[c]

Given a walk $\mathcal{W} = \{w_1, w_2, ..., w_N\}$ of N steps, we denote the total amount of walks possible from g_i to g_j of a graph $\mathcal{G} = \{g_1, g_2, ..., g_M\}$, M finite, in N steps as $\mathcal{W}_N(g_i; g_j)$, with $\mathcal{W}_0(g_i; g_j)$ vanishing for $i \neq j$ and unity for $i = j$. Assuming we can always determine $\mathcal{W}_1$ by simply counting, we can reduce the problem of finding $\mathcal{W}_\mathcal{N}$ into the following summation,

$$\mathcal{W}_N(g_i; g_j) = \sum_{g_l \in \mathcal{G}} \mathcal{W}_{N-1}(g_i; g_l)\mathcal{W}_1(g_l; g_j)$$

$$= \sum_{g_l \in \mathcal{G}} \sum_{g_m \in \mathcal{G}} \mathcal{W}_{N-2}(g_i; g_m)\mathcal{W}_{N-1}(g_m; g_l)\mathcal{W}_1(g_l; g_j)$$

where the second equality demonstrates we can always reduce the first equality into a summation of N $\mathcal{W}_1$'s. To expound on this summation let us consider an example with the graph $\mathcal{G} = \{g_1, g_2, g_3\}$ that constructs a triangle and say we want to determine $\mathcal{W}_3(g_1; g_3)$. Well, by exhaustion we can determine this to be three. But we can also be tricky. We know that $\mathcal{W}_1(g_1; g_3) = \mathcal{W}_1(g_2; g_3) = 1$, $\mathcal{W}_2(g_1; g_1) = 2$ and $\mathcal{W}_2(g_1; g_2) = 1$, with the latter found by minimal exhaustion. What this tells us is that from each walk pertaining to $\mathcal{W}_2(g_1; g_1)$, from $\mathcal{W}_1(g_1; g_3)$ there is only a one 1-step walk to get to g_3 and hence we count two. Similarly, for the other 2-step walk, we count one. Therfore ending with three possible walks.

With the functions $\mathcal{W}_N$ defined, we now introduce a tool that will be extremely useful in determining the probabilities of interest; this tool is called the adjacency matrix.

Definition (Adjacency Matrix). *Given a graph $\mathcal{G} = \{g_1, g_2, ..., g_M\}$ with M finite and $\mathcal{W}_N(g_i; g_j)$ with $g_i, g_j \in \mathcal{G}$, we define the adjacency matrix as a matrix $Adj(\mathcal{G})$ with elements*

$$a_{ij} = \mathcal{W}_1(g_i; g_j).$$

With this definition and the definition of $\mathcal{W}_N$, we have that

$$\mathcal{W}_N(g_i; g_j) = \left[Adj(\mathcal{G})^N\right]_{ij}.$$

But what use is this? Suppose we have a graph $\mathcal{G} = \{g_1, g_2, ..., g_i, ..., g_M\}$ and we want to know the probability that a Markovian stochastic walk starting at g_1 and ending at g_i after N steps. First, we construct $Adj(\mathcal{G})$ by $a_{ij} = \mathcal{W}_1(g_i; g_j)$. Second, we calculate $Adj(\mathcal{G})^N$. Lastly, we simply do the following division

$$P^N_{g_1 g_i} = \frac{\mathcal{W}_N(g_1; g_i)}{\sum_{g_j \in \mathcal{G}} \mathcal{W}_N(g_1; g_j)}.$$

All this division is stating is that the probability from getting from g_1 to g_i is the total number of ways to get from g_1 to g_i divided by the total number of routes of N steps starting at g_1.

Analysis & Theory

We start off our exposition with detailing the probabilities and properties associated with a random walk on a relatively simple graph. From this we dive into null and non-null steps which will allow us to state the Step Saturation theorem. It should be mentioned that some of the results will be supported not by proofs but large data sets; however, it should be mentioned that how one would proof these concepts is given.

Random Walks On A Triangle

Before we dive in, it might be worth making a prediction for motivation. The prediction being thus: Give a graph that describes a triangle and a random walk of N steps on the graph, as N becomes arbitarly large, the probability of finding a particle at any point irregardless of its starting point will tend to $\frac{1}{3}$. One might imagine this plausible because for sufficiently large N it will seem the information of the initial state will be lost in the limit. Let us now dive in.

Consider a graph $\mathcal{G} = \{g_1, g_2, g_3\}$ that constructs a triangle. Then we have the adjacency matrix

$$Adj(\mathcal{G}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

For a walk of $N = 1$ from g_1 to g_2 we should expect $P_{12}^1 = \frac{1}{2}$ since the particle has only two options to move in one step from g_1. We thus have

$$\begin{aligned} P_{12}^1 &= \frac{\mathcal{W}_1(g_1; g_2)}{\sum\limits_{g_j \in \mathcal{G}} \mathcal{W}_1(g_1; g_j)} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2}, \end{aligned}$$

which is what was expected.

What about two steps? Using the matrix method, we have

$$Adj(\mathcal{G})^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

and hence

$$\begin{aligned} P_{12}^2 &= \frac{\mathcal{W}_2(g_1; g_2)}{\sum\limits_{g_j \in \mathcal{G}} \mathcal{W}_2(g_1; g_j)} \\ &= \frac{1}{2+1+1} \\ &= \frac{1}{4}. \end{aligned}$$

We now list results for higher N[2]

N	fraction	decimal
5	$\frac{5}{16}$	0.3125
25	$\frac{5592405}{16777216}$	~ 0.3333333135
50	$\sim \frac{1.87 \cdot 10^{14}}{5.629 \cdot 10^{14}}$	$\sim 0.333...$
75	$\sim \frac{1}{3}$	$\sim 0.333...$
100	$\sim \frac{1}{3}$	$\sim 0.333...$

We see even at $N = 25$ the probability is approaching $\frac{1}{3}$. At higher N, the values are even closer. This supports the prediction we made earlier. We will discuss how to possibly prove it in the **Future Directions Of Study** section.

We shall now demonstrate this process for more complicated structure.

Random Walks On A Cube

Consider a graph $\mathcal{G} = \{g_1, g_2, ..., g_8\}$ that constructs a cube. For sake of being succint, we shall compute $Adj(\mathcal{G})$ to implicitly detail which points are connected to which. We get for the matrix

$$Adj(\mathcal{G}) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Notice we have the symmetry and number of nonzero entries in each row as we would expect–each point can connect to at most three points.

Using the same equation as we did for the triangle we find, as an example, for $N = 6$, we have a probability of $P^6_{g_1 g_8} = \frac{20}{81}$. There is something interesting that happens for other values of N. We find that for odd values of N we have a vanishing probability. The reasoning for this will be understood after we discuss null structures.

Non-null Structure

By structure we will mean a subgraph with a specific characteristic. The first structure we will meet is the non-null structure.

Definition (Non-null Structure). *Given a graph $\mathcal{G} = \{g_1, g_2, ..., g_M\}$, if there exists a subgraph $\mathcal{G}^{\mathcal{N}} \subseteq \mathcal{G}$ such that $P^{\mathcal{N}}_{g_i g_j} > 0$ for any $g_j \in \mathcal{G}^{\mathcal{N}}(g_l, ..., g_m)$ and $g_i \in \mathcal{G}$ for some walk $\mathcal{W} = \{w_1, w_2, ..., w_{\mathcal{N}}\}$, we call the subgraph a non-null structure of order $\mathcal{N}$ and liberator on points $g_l, ..., g_m$ because it liberates $P^{\mathcal{N}}_{g_i g_j}$ from being restriced to zero for $g_j \in \{g_l, ..., g_m\}$.*

[2]These values were determined using TI-BASIC on a TI-83; refer to end of paper to see code used.

We have already encountered a non-null structure; namely the graph defining the triangle. To see this non-null structure in a more interesting case, consider the tetrahedron. For large enough N, the particle can always circle around the triangle made by the remaining points not desired to end on for $N-1$ steps. But because each point of that triangle, i.e., the non-null structure, are connected to the desired point, the particle can land on the desired point at the Nth step. This leads to the following theorem.

Theorem (Step Saturation). *Suppose we have a graph $\mathcal{G} = \{g_1, g_2, ..., g_M\}$ and a non-null structure $\mathcal{G}^{\mathcal{N}}$ of $\mathcal{G}$. Then for large enough N for a walk of N steps, it will always be possible to land on any point on $\mathcal{G}^{\mathcal{N}}$ or some points attached to $\mathcal{G}^{\mathcal{N}}$.*

Proof. Suppose we have a graph $\mathcal{G} = \{g_1, g_2, ..., g_M\}$ and a non-null structure $\mathcal{G}^{\mathcal{N}}$ of $\mathcal{G}$. Suppose further we are considering a N-step walk from g_1 to g_i where $g_i \in \mathcal{G}^{\mathcal{N}}$. If we let M be large enough so that $\mathcal{W}_M(g_1; g_i) \neq 0$ and have $N - M > \mathcal{N}$, then, by the nature of the non-null structure, it will always be possible to land on a point of the structure. Furthermore, if we let N be large enough so that at $N-1$ we land on a point g of $\mathcal{G}^{\mathcal{N}}$, which, by definition of non-null structure, we can, we will be able to land on any point connected to g at the Nth step. Therefore, both cases of the theorem have been proven.

Definition (Null Structure). *Given a graph $\mathcal{G} = \{g_1, g_2, ..., g_M\}$, if there exists a subgraph $\mathcal{G}^{\mathcal{N}} \subseteq \mathcal{G}$ such that $P^{\mathcal{N}}_{g_i g_j} = 0$ for any $g_j \in \mathcal{G}^{\mathcal{N}}(g_l, ..., g_m)$ and $g_i \in \mathcal{G}$ for some walk $\mathcal{W} = \{w_1, w_2, ..., w_{\mathcal{N}}\}$, we call the subgraph a null structure of order $\mathcal{N}$ and restrictor on points $g_l, ..., g_m$ because it restricts $P^{\mathcal{N}}_{g_i g_j}$ to zero for $g_j \in \{g_l, ..., g_m\}$.*

We can understand now why the probabilities of the cube above were zero; namely, there exists a null structure in the graph that defined the cube. In fact, it turns out that the null structure is a square.

Theorem (Square Null Structure). *Suppose we have a graph $\mathcal{G} = \{g_1, g_2, g_3, g_4\}$ that defines a square in a sensical way. Then there are values of N such that the probability $P^N_{g_i g_j} = 0$ where g_i and g_j are not connected.*

Proof. Suppose we have a graph $\mathcal{G} = \{g_1, g_2, g_3, g_4\}$ defining a square such that g_1 and g_3 are not connected. Then we have $\mathcal{W}_1(g_1; g_3) = 0$ since it takes at least two steps to get from g_1 to g_3. It turns out that it will always take an even number N of steps to get from g_1 to g_3 because from

$$\mathcal{W}_N(g_i; g_j) = \sum_{g_l \in \mathcal{G}} \mathcal{W}_{N-1}(g_i; g_l)\mathcal{W}_1(g_l; g_j)$$

we get

$$\begin{aligned}
\mathcal{W}_N(g_1; g_3) &= \sum_{g_l \in \mathcal{G}} \mathcal{W}_{N-1}(g_1; g_l)\mathcal{W}_1(g_l; g_3) \\
&= \mathcal{W}_{N-1}(g_1; g_2) + \mathcal{W}_{N-1}(g_1; g_4) \\
&= \mathcal{W}_{N-2}(g_1; g_1) + \mathcal{W}_{N-2}(g_1; g_3) + \mathcal{W}_{N-2}(g_1; g_1) + \mathcal{W}_{N-2}(g_1; g_3) \\
&= 2\mathcal{W}_{N-2}(g_1; g_1) + 2\mathcal{W}_{N-2}(g_1; g_3)
\end{aligned}$$

where we used the fact that $\mathcal{W}_1(g_i; g_i) = 0$ for all i and $\mathcal{W}_1 = 1$ for all g_i and g_j connected. To finish the proof we will use a contradiction. From

$$\mathcal{W}_N(g_1; g_3) = 2\mathcal{W}_{N-2}(g_1; g_1) + 2\mathcal{W}_{N-2}(g_1; g_3)$$

we just derived, assume N is odd. Then $N-2$ is odd and hence $\mathcal{W}_{N-2}(g_1; g_1) = 0$ since it will always take a step away and a step back or four steps around to get from g_1 and back; that is to say, $N-2$ must be even for $\mathcal{W}_{N-2}(g_1; g_1) \neq 0$. It then follows

$$\mathcal{W}_N(g_1; g_3) = 2\mathcal{W}_{N-2}(g_1; g_3).$$

It is then immediate that for N odd, $\mathcal{W}_N = 0$ since at $N = 3$, we have

$$\mathcal{W}_3(g_1; g_3) = 2\mathcal{W}_1(g_1; g_3) = 0$$

and hence by the equation being recursive, all N odd have $\mathcal{W}_N = 0$.

Future Directions Of Study

Though the equation

$$P^N_{g_1 g_i} = \frac{\mathcal{W}_N(g_1; g_i)}{\sum\limits_{g_j \in \mathcal{G}} \mathcal{W}_N(g_1; g_j)}$$

is sufficient for finding probabilities, the number of computations for large N can get quite large. This is where I posit the possibility of using the Chapman-Kolmogorov equation and appropriate symmetries in summation. As shown above, it is possible to reduce the Chapman-Kolmogorov equation a summation of summations of, etc. Noting that each summation is summing over the same statespace, it should be conceivable that one could, using symmetry in the summations, be able to reduce the problem to a single summation. Assuming this to be possible, if the probability of interest is finite, using summation techniques, one could solve for an explicit equation that defines the probability desired.

Non-Markovian Walks

There is no reason to restrict the random walk to having the Markovian property. For example, suppose that once the particle lands on a point, the porbability of a particle landing on it again vanishes. In this case, we wouldn't need to restrict the walk to N steps because since after each point is reach it's probability of being reached again vanishes, only a finite amount of points will be able to be reached; that is to say, the defining property of this kind of walk limits the number of steps for us.

Two Particle Walk

Suppose instead of a single particle walking we have two. There are several probabilities we can seek in this case. One would be asking What is the probability of the two particles meeting at N steps. Another would be be asking What is the probability of the two particles meeting at a specific point. There are other variations that can be done.

Other Structures

Considering other possible structures that are null or non-null, using the proof for the square structure, one can see it pluasible to use a similar proof for other null structures. Similarly, the proof for the non-null structure can be used for other non-null structures. Once other structures are found, the Step

Saturation and Step Restriction theorem can be used more powerfully.

Conclusion

After developing some basic theory of the fields of stochastic proceses and graph theory, we have seen how to calculate the probability for a particle traveling to and from two points on a finite graph whereby the particle's travel abides the Markovian property and is random. Furthermore, we explored two interested subgraph structures that either restricted or liberated the probability from vanishish. As was seen, these structures could be used to simplify the process of determining whether or not the probability for traveling from some point to another would vanish based on whether or the point was a part of some structure.

TI-BASIC Code

The follow code was used to determine the probabilities associated with the random walk of steps $N = 1$ to $N = 100$ on a triangle from point g_1 to g_2.

$\{1\} \to L_1$
$\{1\} \to L_2$
$\{1\} \to L_3$
$For(A, 1, 100, 1$
$[A]\hat{}A \to [B]$
$augment(L_1, \{[B](1,2)\}) \to L_1$
$augment(L_2, \{A\}) \to L_2$
$augment(L_3, \{[B](1,1)\} + [B](1,2)\} + [B](1,3)\}) \to L_3$
End
$\{1\} \to L_4$
$For(X, 2, dim(L_1), 1$
$augment(L_4, \{L_1(x)/L_3(x)\}) \to L_4$
End

Bibliography

[a] Stochastic Processes - Ross QA 274 R65 1983

[b] http://www.hamilton.ie/ollie/

[c] Graph Theory and Theoretical Physics, Frank Harary, 1967

[d] An Introduction to Stochastic Processes in Physics, Don S. Lemons, 2002