**Fourier Transform**

Math326 Independent Project

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A lot of things in the world can be described via a waveform - a function of time, space or some other variable. For instance, sound waves, electromagnetic fields, music you listen from radio, the price of your favorite stock versus time, the frequency of your breath, etc. Some of them we can see directly by recording as a graph, but some we can’t. Fourier Transform is plays an important role as gives us a unique and powerful way of viewing these waveforms directly. The Fourier Transform is used in a wide range of applications, such as image analysis, image filtering, image reconstruction and image compression.

A French mathematician and physicist Jean Baptiste Joseph Fourier named Fourier analysis. Joseph Fourier made the first systematic use, although not a completely rigorous investigation. The paper had the controversial claim that any continuous periodic signal could be represented as the sum of properly chosen sinusoidal waves. One reviewer of this paper, the mathematician Lagrange insisted that there was an approach could not be used to represent signals with corners such as discontinuous slopes. Lagrange’s view was correct but not exactly, since the difference between the two sinusoids that have zero energy was so close. The paper finally published only after Lagrange dead. Although it turned out that Fourier’s claim of generality was somewhat too strong, his results stimulated a flood of important research that has continued to the present day.

Fourier transform is a mathematical transform with many applications in physics and engineering. We can solve many important problems involving partial differential equations by using trigonometric series that is called Fourier series. Fourier series is one part of Fourier transform that we already talk about in Math 309. Except Fourier series, another three are Fourier transform (one type of Fourier transform), Discrete Time Fourier Transform and Discrete Fourier Transform.

 Before we gain insight into these interesting topics, we need to know some background use in the Fourier transform, such as the Dirac delta function (also name as impulse function) and complex number. Impulse function is the limit of a sequence of functions (Function 1) on the real number line that is zero everywhere except at zero, with an integral of one over the entire real line. The complex number also will use in Fourier transform. (Function 2)

 (Function 1)

 (Function 2)

 After the background, the first thing need to show is how to define the Fourier Transform. At the most of time, the Fourier Transform of a function g(t) is defined by : . f is the frequency, G(f) is trying to show that how much power g(t) contains at the frequency f. Moreover, g can be obtain from G by the inverse Fourier Transform:

Hence, we get the g(t) from G(f) by using inverse Fourier Transform. Therefore we call g(t) and G(f) a Fourier Pair.

Now it’s time to talk about the “body” of Fourier Transform. As we know, the Fourier Transform is an important image-processing tool which can used in decomposing an image into its sine and cosine components. Images are all from signals. A signal can be either continuous or discrete, and also can be either periodic or aperiodic. Therefore Fourier transform can be separated into four types, which are continuous periodic, discrete periodic, continuous aperiodic and discrete aperiodic.

* Periodic Continuous (Fourier Series)

The signal repeats itself in a periodic pattern from negative to positive infinity.

* Periodic Discrete (Discrete Fourier Transform)

The discrete signal repeats itself in a periodic pattern from negative to positive infinity.

* Aperiodic Continuous (Fourier Transform)

The signal extends to both negative and positive infinity without repeating itself in a periodic pattern.

* Aperiodic Discrete (Discrete Time Fourier Transform)

The discrete signal extends to both negative and positive infinity without repeating itself in periodic patter.

|  |  |
| --- | --- |
| Type of Transform | Example Signal |
| Periodic Continuous- Fourier Series |  |
| Periodic Discrete-Discrete Fourier Transform |  |
| Aperiodic Continuous-Fourier Transform |  |
| Aperiodic Discrete-Discrete Time Fourier Transform |  |

The table (Steven) above shows four examples of signals and the difference of these four type Fourier transform.

1. Fourier Series

In the mathematics 309 courses we first touched the problems about Fourier Transform and learnt Fourier series. The Fourier series can be view as a special introductory case of the Fourier Transform and Fourier series is an important basic things need to know, therefore no Fourier Transform tutorial is complete without a study of Fourier series. Euler-Fourier formulas can be simplified into two different functions. These are even and odd functions, which are characterized geometrically by the property of symmetry with respect to the y-axis and the origin.

(Boyce)

 (a) An even function (b) An odd function

From these two graph above, we can see that in the (a) is an even function f (-x) = f (x), and in the (b) an odd function f (-x) = - f (x). Therefore we need to consider the Fourier series in two different directions such as cosine and sine.

In the cosine series, if we suppose that f and f’ are continuous on −L ≤ x < L and therefore f is an even periodic function with period 2L. Then f (x) cos(nπ x/L) will be an even function and f (x) sin(nπ x/L) will be an odd function, the Fourier coefficients of f are then given by

Thus f has the Fourier series

In the sine series, if we suppose that f and f’ are continuous on −L ≤ x < L and therefore f is an odd periodic function with period 2L. Then f (x) cos(nπ x/L) will be an odd function and f (x) sin(nπ x/L) will be an even function, the Fourier coefficients of f are then given by

Thus f has the Fourier series

We use Fourier series to do lot of questions before, such as the heat conduction problem, wave equation, vibrations of an elastic string, etc.

1. Discrete Fourier Transform (DFT)

"Time domain" in Fourier analysis, it may actually refer to samples taken over time, or it might be a general reference to any discrete signal that is being decomposed. The term frequency domain is used to describe the amplitudes of the sine and cosine waves. Unlike the other three Fourier Transforms, both views of time domain and frequency domain in discrete Fourier transform are periodic. The [sequence](http://en.wikipedia.org/wiki/Sequence) of N [complex numbers](http://en.wikipedia.org/wiki/Complex_number) x0, x1… xn-1 is transformed into an N-periodic sequence of complex numbers x0, x1, ….., xn-1 according to the DFT formula:

In this context, it is common to define  to be the Nth primitive [root of unity](http://en.wikipedia.org/wiki/Root_of_unity), , to obtain the following form:

1. Fourier Transform (FT)

The time domain signal extends from negative to positive infinity, while each of the frequency domain signals extends from zero to positive infinity. This frequency spectrum is shown in rectangular form (real and imaginary parts); however, the polar form (magnitude and phase) is also used with continuous signals. Just as in the discrete case, the synthesis equation describes a recipe for constructing the time domain signal using the data in the frequency domain. The function can be defined by:

1. Discrete Time Fourier Transforms (DTFT)

The Discrete Time Fourier Transform (DTFT) is one of the Fourier transforms family that operates on aperiodic and discrete signals. DTFT can be easily understood if you relate it to the DFT. For example, suppose you acquire an N sample signal, and want to find the frequency spectrum of these signals. At first if you are using the DFT, the signal will be decomposed into sine and cosine two waves, with frequencies equally spaced between zero and one-half of the sampling rate. The interesting thing is when N approaches infinity, the time domain becomes aperiodic, and the frequency domain becomes a continuous signal. That is the DTFT, the Fourier transform that relates an aperiodic, discrete signal, with a periodic, continuous frequency spectrum.

Given a discrete set of real or complex numbers: x[n], for all integers, the discrete-time Fourier transform (or DTFT) of x[n] is usually written:

 In conclusion, these four types of Fourier transforms are different, but they are really close and relative to each other. After defining the signal periodic or aperiodic, continuous or discontinuous, and thinking about the time domain and frequency domain, then we can use the function and try to do Fourier transforms.

 Another simpler way to do with Fourier transform is use the Matlab. MATLAB  (matrix laboratory) is a [numerical computing](http://en.wikipedia.org/wiki/numerical_analysis) environment and [fourth generation programming language](http://en.wikipedia.org/wiki/fourth-generation_programming_language). The way using Matlab also called a fast Fourier transform, which is more efficiency and high productivity by using technique. As well, the Fourier transform also can be defined by lots form in different type problems such as heat problem, which are the things we need to solve in the future.

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